THE PARTIAL LEGENDRE TRANSFORM IN MONGE-AMPÈRE EQUATIONS

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Abstract. This survey explores the application of the partial Legendre transform in the context of Monge-Ampère equations, based on recent work by the authors [WZ1, WZ2, Wa]. Specifically, we demonstrate how the partial Legendre transform can be utilized to establish interior estimates and Liouville-type theorems for Monge-Ampère equations, linearized Monge-Ampère equations, as well as Monge-Ampère type fourth order equations in two dimensions.

1. Introduction

The Monge-Ampère equation is a fundamental fully nonlinear partial differential equation with important applications in differential geometry, optimal transport, and geometric analysis, which draws its name from its initial formulation in two dimensions, by the French mathematicians Monge [Mo] and Ampère [Am], about 200 years ago. A useful tool for studying Monge-Ampère equations is the *Legendre transform* (or *Legendre transformation*), first introduced by Legendre in 1787 in his study of minimal surfaces [Leg]. The Legendre transform is an involutive transformation. Specifically, for an open set $\Omega \subset \mathbb{R}^n$ and a function $u : \Omega \to \mathbb{R}$, the Legendre transform $u^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of u is defined as

$$u^*(\xi) = \sup_{x\in\Omega} \left(x \cdot \xi - u(x)\right).$$

It is easy to see that when u is convex, the supremum is attained at the point x where $\xi = Du(x)$, assuming u is sufficiently smooth, i.e. $u^*(\xi) = x \cdot \xi - u(x)$, see geometric interpretation in Figure 1. From this, we deduce that $(u^*)^* = u$, confirming that the Legendre transform is an involutive transformation. Consequently, the following relations hold:

$$Du(x) = \xi, \quad \text{and} \quad Du^*(\xi) = x.$$

Differentiating further yields

$$D^2u(x)D^2u^*(\xi) = Id.$$

By taking the determinant of both sides, if $\det D^2u = f$, then

$$\det D^2u^* = \frac{1}{f \circ Du^*},$$

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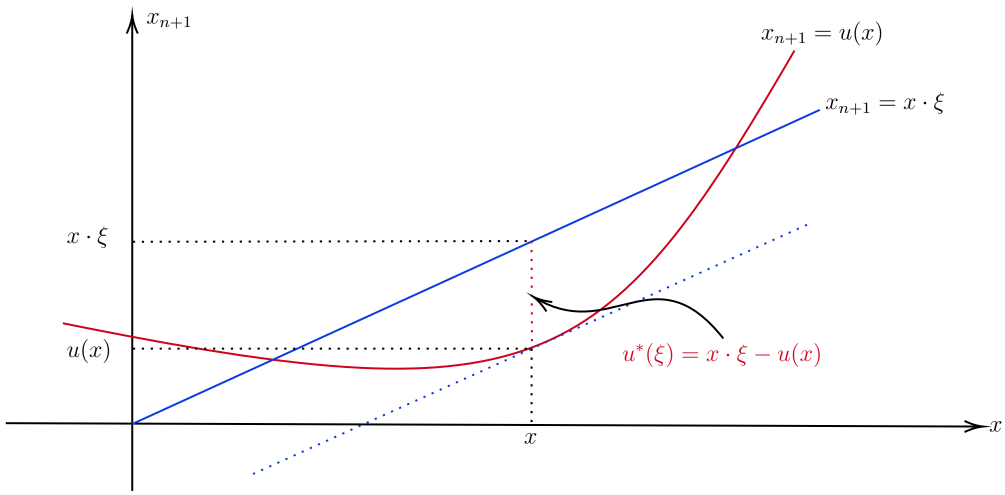

FIGURE 1. The Legendre Transform

This indicates that the Legendre transform exchanges the upper and lower bounds on the Hessian's determinant, so typically, finding one bound on the determinant is sufficient, as the other bound can be derived similarly after applying the Legendre transform. Furthermore, if $f = K(1 + |Du|^2)^{\frac{n+2}{2}}$ with $K > 0$, i.e, the constant Gaussian curvature equation, then u^* satisfies

$$\det D^2 u^* = \frac{(1 + |\xi|^2)^{-\frac{n+2}{2}}}{K}.$$

Note the right-hand side term is independent of u^*. Thus, the Legendre transform can sometimes simplify the equation.

The Legendre transform is widely used in the study of Monge-Ampère equations (see [Fi, Gu, TW2] and references therein). It should be noted that the transformation preserves the fully nonlinear nature of these equations, making them somehow still challenging to analyze. Hence, ideally, one might seek a transformation that simplifies the Monge-Ampère equation to a linear form, allowing the application of the theory for linear equations. Such a transformation is only accessible in two dimensions, and is known as the *partial Legendre transform*, introduced by Darboux in the early 19th century, and is the main focus of this survey. Precisely, in this transformation, we apply the Legendre transform to a single variable, which can transform the Monge-Ampère equation into a quasilinear equation. In the following, we show this process explicitly.

Let $\Omega \subset \mathbb{R}^2$, and let $u(x, y)$ be a convex function defined on Ω, with subscripts used to denote partial derivatives (e.g., $u_x = \frac{\partial u}{\partial x}$, etc.). The partial Legendre transform in the x-variable is given by

$$u^\star(\xi, \eta) = \sup \left(x\xi - u(x, \eta) \right),$$

where the supremum is taken with respect to x for fixed η, i.e., for all x such that $(x,\eta)\in\Omega$. This definition is from [Liu]. When $u\in C^2(\Omega)$ is strictly convex in the x-variable, the mapping
$$\mathcal{P}:(x,y)\mapsto(u_x(x,y),y)$$
is injective, and we denote the image of Ω under $\mathcal{P}$ as $\Omega^\star$. In this case, we have
$$u^\star(\xi,\eta)=xu_x(x,y)-u(x,y)\quad\text{in}\quad\Omega^\star.$$
This follows directly from the strict convexity of u with respect to the x-variable [GP].

A direct calculation yields the following derivatives of the partial Legendre transform:
$$\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{pmatrix}u_{xx} & u_{xy}\\ 0 & 1\end{pmatrix},\quad \frac{\partial(x,y)}{\partial(\xi,\eta)}=\begin{pmatrix}\frac{1}{u_{xx}} & -\frac{u_{xy}}{u_{xx}}\\ 0 & 1\end{pmatrix}.$$
From this, we obtain the relations
$$u^\star_\xi=x,\quad u^\star_\eta=-u_y, \tag{1.1}$$
and
$$u^\star_{\xi\xi}=\frac{1}{u_{xx}},\quad u^\star_{\eta\eta}=-\frac{\det D^2u}{u_{xx}},\quad u^\star_{\xi\eta}=-\frac{u_{xy}}{u_{xx}}. \tag{1.2}$$
Now, if $u\in C^2(\Omega)$ is a strictly convex solution to the equation
$$\det D^2u=f(x,y,u,u_x,u_y),$$
then $u^\star\in C^2(\Omega^\star)$ is a solution to the transformed equation
$$f(u^\star_\xi,\eta,\xi u^\star_\xi-u^\star,\xi,-u^\star_\eta)u^\star_{\xi\xi}+u^\star_{\eta\eta}=0. \tag{1.3}$$
This reduction is significant because it allows the application of tools and techniques typically used in the study of quasilinear equations, making the analysis more manageable in this specific context. The partial Legendre transform has been employed by many authors to convert the two-dimensional Monge-Ampère equation into a quasilinear elliptic equation, demonstrating its effectiveness as a powerful tool, especially for some degenerate Monge-Ampère equations (see, for example, [DS, Fi, Gua, GP, GS, LS, Liu, RSW1, RSW2, SW1, SW2, Sch]). In addition, Rubin used the partial Legendre transform in [Ru] to study the Monge-Ampère equation with Guillemin boundary conditions. While there are many applications of the partial Legendre transform, we cannot list all of them here. We refer readers to the references above and those cited therein.

In this survey, we will demonstrate how the simplification facilitated by the partial Legendre transform makes it a powerful tool in the study of Monge-Ampère equations, linearized Monge-Ampère equations, as well as Monge-Ampère type fourth order equations in two dimensions. The organization is as follows: In Section 2, we present several Liouville theorems for a class of Monge-Ampère equations. Section 3 is dedicated to deriving interior estimates for Monge-Ampère type fourth order equations. Finally, in Section 4, we use the partial Legendre transform to obtain interior Hölder estimates for linearized Monge-Ampère equations.

2. Liouville theorems for a class of Monge-Ampère equations

In this section, we show how the partial Legendre transform can be used to prove Liouville type theorems for Monge-Ampère equations in two dimensions.

First, by (1.3), if $u(x,y) \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ is a convex solution to

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^2_+, \quad u = \frac{x^2}{2} \quad \text{on } \partial\mathbb{R}^2_+,$$

then its partial Legendre transform $u^\star(\xi,\eta) \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ satisfies

$$\Delta u^\star = 0 \quad \text{in } \mathbb{R}^2_+, \quad u^\star = \frac{\xi^2}{2} \quad \text{on } \partial\mathbb{R}^2_+.$$

By the classical Schauder estimates for the Laplace equation, we see that $u^\star \in C^\infty(\overline{\mathbb{R}^2_+})$. We can then differentiate $u^\star$ twice with respect to ξ to obtain

$$\Delta u^\star_{\xi\xi} = 0 \quad \text{in } \mathbb{R}^2_+, \quad u^\star_{\xi\xi} = 1 \quad \text{on } \partial\mathbb{R}^2_+, \tag{2.1}$$

and $u^\star_{\xi\xi} \geq 0$ (the partial Legendre transform preserves convexity in the ξ direction). Using a logarithmic barrier, $w(\xi,\eta) := -\ln(\sqrt{\xi^2 + (\eta+1)^2})$, we find that $v := u^\star_{\xi\xi} - 1$ is nonnegative, and thus satisfies

$$\Delta v = 0 \quad \text{in } \mathbb{R}^2_+, \quad v = 0 \quad \text{on } \partial\mathbb{R}^2_+, \quad \text{and} \quad v \geq 0 \quad \text{in } \mathbb{R}^2_+.$$

It is a classical result that solutions to this problem must take the form $v(\xi,\eta) = A\eta$ for some $A \geq 0$ (see, for example, [BB, Theorem 1]). Therefore, $u^\star_{\xi\xi} = 1 + A\eta$. Solving the some related ODEs, we obtain

$$u^\star(\xi,\eta) = B\eta - \frac{\eta^2}{2} - \frac{A\eta^3}{6} + C\xi\eta + \frac{\xi^2}{2}(1 + A\eta)$$

for some constants $B, C \in \mathbb{R}$. Since the Legendre transform is an involution on convex functions, we recover u by taking the partial Legendre transform of $u^\star$ in the ξ direction, which gives us that

$$u(x,y) = \frac{y^2}{2} + \frac{Ay^3}{6} - By + \frac{(x - Cy)^2}{2(1 + Ay)}.$$

To summarize, we have the following theorem:

Theorem 2.1 ([Fi, Page 147-148]). *Let $u(x,y) \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ be a convex solution to*

$$\begin{cases} \det D^2 u = 1 & \text{in } \mathbb{R}^2_+, \\ u(x,0) = \dfrac{x^2}{2} & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

Then there exist constants $A, B, C \in \mathbb{R}$ with $A \geq 0$ such that

$$u(x,y) = \frac{y^2}{2} + \frac{Ay^3}{6} - By + \frac{(x - Cy)^2}{2(1 + Ay)}.$$

Using the partial Legendre transform, we classify the following class of Monge-Ampère equations in [WZ2]

$$\det D^2u = (a+by)^{\alpha}, \quad \alpha > -2 \tag{2.2}$$

in the half space $\mathbb{R}^2_+$, where $a \geq 0$ and $b > 0$. A motivation to consider (2.2) comes from the study of the Monge-Ampère type fourth order equations (3.1), will also be discussed in the next section. To study a Liouville type theorem for (3.1) under the boundary condition

$$\begin{cases} u = \dfrac{1}{2}|x'|^2 & \text{on } \partial\mathbb{R}^2_+, \\ w = 1 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.3}$$

one can easily find that solutions to (2.2) with $\alpha = -\frac{1}{\theta}$, $a = b = 1$ give a class of special solutions to (3.1), (2.3). The classification of all solutions to (2.2) can help us to study (3.1), (2.3). When $a = 0$, the higher dimension case for (2.2) were studied by [Sa1, Sa2, SZ]. Here we only focus on the two dimensional case.

After performing the partial Legendre transform, u^* satisfies the following Grushin type equation

$$(a+b\eta)^{\alpha}u^{\star}_{\xi\xi} + u^{\star}_{\eta\eta} = 0.$$

Differentiating twice with respect to ξ, we obtain an equation similar to (2.1):

$$(a+b\eta)^{\alpha}(u^{\star}_{\xi\xi})_{\xi\xi} + (u^{\star}_{\xi\xi})_{\eta\eta} = 0. \tag{2.4}$$

Thus, to complete the argument, it suffices to use a Liouville theorem, similar to [BB, Theorem 1], for (2.4) to show that $u^{\star}_{\xi\xi} = 1 + A\eta$. However, directly proving a Liouville theorem for (2.4) is somewhat challenging. Fortunately, we find that by performing a change of variables

$$v(x_1, x_2) = u^{\star}_{\xi\xi}(x_1, f(x_2)),$$

where

$$\xi = x_1, \quad \eta = f(x_2) = b^{\frac{-\alpha}{\alpha+2}}\left(\frac{\alpha+2}{2}x_2\right)^{\frac{2}{\alpha+2}} - \frac{a}{b},$$

v satisfies the following divergence-type equation:

$$\operatorname{div}\left(x_2^{\frac{\alpha}{\alpha+2}}\nabla v\right) = 0,$$

for which the Liouville theorem can be derived using the method of moving spheres. The details of this method are referred to [WZ2, Theorem 2.1], which gives us the desired Liouville theorem for v. Hence, we conclude that:

Theorem 2.2 ([WZ2, Theorem 1.1]). *Let $u(x,y) \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ be a convex solution to*

$$\begin{cases} \det D^2u = (a+by)^{\alpha} & \text{in } \mathbb{R}^2_+, \\ u(x,0) = \dfrac{1}{2}x^2 & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

where $a \geq 0$, $b > 0$, and $\alpha > -2$. Then there exist A, B, $C \in \mathbb{R}$ with $A \geq 0$ such that

$$u(x,y) = \begin{cases} \dfrac{(b-aA)(a+by)^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} + \dfrac{A(a+by)^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} - By \\ \qquad - \dfrac{(b-aA)a^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} - \dfrac{Aa^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} + \dfrac{(x-Cy)^2}{2(1+Ay)}, & \alpha \neq -1; \\ \dfrac{b-aA}{b^3}(a+by)\ln(a+by) + \dfrac{A}{2b}y^2 - By \\ \qquad\qquad - \dfrac{(b-aA)a\ln a}{b^3} + \dfrac{(x-Cy)^2}{2(1+Ay)}, & \alpha = -1. \end{cases}$$

Interestingly, the above approach also applies to the case of the Neumann problem. By using nearly identical arguments, we obtain the following result:

Theorem 2.3 ([WZ2, Theorem 1.3])**.** *Let $u(x,y) \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ be a convex solution to*

$$\begin{cases} \det D^2 u = y^\alpha & \text{in } \mathbb{R}^2_+, \\ u_y(x,0) = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

where $\alpha > -1$. Then there exist a constant $A > 0$, and a linear function $l(x)$ such that

$$u(x,y) = \frac{1}{2A}x^2 + \frac{A}{(2+\alpha)(1+\alpha)}y^{2+\alpha} + l(x).$$

3. Interior estimates for Monge-Ampère type fourth order equations

Using the partial Legendre transform to derive interior estimates for Monge-Ampère equations in two dimensions is well known, as seen in [DS, Gua, Liu], and other references therein. However, the first use of the partial Legendre transform to study Monge-Ampère type fourth order equations appears in [LZ] by Le and the second author, where it played a crucial role in solving the second boundary value problem for singular Abreu equations in two dimensions. Motivated by this approach, we consider a slightly more general class of fourth order equations in [WZ1], which includes affine mean curvature equations and Abreu's equation. Let $\Omega \subset \mathbb{R}^2$ be a convex domain, and we study the regularity of the following fourth order equations of Monge-Ampère type:

$$\sum_{i,j=1}^{2} U^{ij} w_{ij} = 0, \tag{3.1}$$

in Ω, where $\{U^{ij}\}$ is the cofactor matrix of D^2u of an unknown uniformly convex function, and

$$w = \begin{cases} [\det D^2 u]^{-(1-\theta)}, & \theta \in [0,1), \\ \log\det D^2 u, & \theta = 1. \end{cases} \tag{3.2}$$

When $\theta = \frac{1}{n+2}$, it is the *affine mean curvature equation* in affine geometry [Ch]. When $\theta = 0$, it is *Abreu's equation* arise from the problem of extremal metrics on toric manifolds in Kähler geometry [Ab].

After applying the partial Legendre transform, the Monge-Ampère type fourth order equation (3.1) reduces to a quasilinear second order equation for the determinant (3.3). This can be achieved either by directly calculating all order derivatives after the transform or by deriving the Euler-Lagrange equation for the transformed functional of (3.1). Specifically, we have:

Proposition 3.1 ([WZ1, Proposition 2.1])**.** *Let u be a uniformly convex solution to* (3.1) *in Ω. Then in $\Omega^\star = \mathcal{P}(\Omega)$, its partial Legendre transform $u^\star$ satisfies*

$$w^\star w^\star_{\xi\xi} + w^\star_{\eta\eta} + (\theta-1)w^{\star 2}_{\xi} + \frac{\theta-2}{w^\star}w^{\star 2}_{\eta} = 0. \tag{3.3}$$

Here $w^\star = -\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}$.

This simplification allows us to obtain interior estimates for solutions to the fourth order equations by deriving interior estimates for solutions to the corresponding quasilinear second order equations, where classical techniques for second order elliptic equations can be applied, and this is also the key ingredient of [WZ1].

Theorem 3.2 ([WZ1, Theorem 2.2])**.** *Assume $w^\star$ is a solution to* (3.3) *with $\theta \in [0,1]$ on $B_R := B_R(0)$ and satisfies $0 < \lambda \le w^\star \le \Lambda$. Then there exist $\alpha, C > 0$ depending on λ, Λ, R and θ, such that*

$$\int_{B_R} |Dw^\star|^3 (R^2 - \xi^2 - \eta^2)^\alpha \, \mathrm{d}V \le C. \tag{3.4}$$

The proof of Theorem 3.2 is quite standard. First, we introduce an auxiliary function $z = v\phi\eta$, where

$$v = \sqrt{|Dw^\star|^2 + 1},\ \eta = (R^2 - \xi^2 - \eta^2)^\alpha \text{ with } \alpha > 3,$$

and

$$\phi(w^\star) = Aw^{\star\theta-2} - \frac{w^\star}{2\theta^2 - 9\theta + 9} \text{ with } A \ge \frac{\Lambda^{3-\theta}}{2\theta^2 - 9\theta + 9} + 1.$$

Next, we calculate the equation that z satisfies. Finally, after a complex and technical calculation, we obtain (3.4). Now, it is enough to obtain interior estimates for fourth order equations. We proceed as follows:

For any $x \in \Omega$, we denote $R = \frac{\mathrm{dist}(x,\partial\Omega)}{2}$. Without loss of generality, we assume $\mathcal{P}(x) = 0$. If we assume u satisfies $\lambda \le \det D^2 u \le \Lambda$, hence we know that by [Liu, Lemma 2.1], there exists $\delta > 0$ depending on $C(R)$ such that $B_\delta(0) \subset \mathcal{P}(B_R(x))$. With Theorem 3.2 in hand, we obtain

$$\|w^\star\|_{W^{1,3}(B_{7\delta/8}(0))} \le C.$$

Note that here $n = 2$. By the Sobolev theorem, we have the C^α estimate of $w^\star$. And by the interior $W^{2,p}$-estimate of the uniformly elliptic equation (3.3), we have the estimate

$$\|w^\star\|_{W^{2,3/2}(B_{3\delta/4}(0))} \leq C,$$

which implies the $W^{1,6}$-estimate of $w^\star$. Again by the interior $W^{2,p}$-estimate of the uniformly elliptic equation (3.3), we have the

$$\|w^\star\|_{W^{2,3}(B_{\delta/2}(0))} \leq C,$$

which implies the $C^{1,\alpha}$ estimate of $w^\star$. Then by the Schauder estimate of (3.3), we have

$$\|w^\star\|_{C^{2,\alpha}(B_{\delta/4}(0))} \leq C$$

and all the higher order estimates of $u^\star$. Transforming back by the partial Legendre transform, we obtain the lower bound of $u_{x_1x_1}$ by (1.2). Since we can do partial Legendre transforms of u in any direction, we can obtain the lower bound for the smallest eigenvalue of D^2u, which implies the boundedness of D^2u by (3.5). Then we have all the higher order estimates of u. Hence, we have shown:

Theorem 3.3 ([WZ1, Theorem 1.1])**.** *Assume $n = 2$ and $\theta \in [0, 1]$. Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let u be a smooth convex solution to equation* (3.1) *on Ω satisfying*

$$0 < \lambda \leq \det D^2u \leq \Lambda. \tag{3.5}$$

Then for any $\Omega' \subset\subset \Omega$, there exists a constant $C > 0$ depending on $\sup_\Omega |u|$, λ, Λ, θ and $\mathrm{dist}(\Omega', \partial\Omega)$*, such that*

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C.$$

By Theorem 3.3 and a rescaling argument as in [TW1, Theorem 2.1], we obtain a new proof of the following Bernstein theorem [TW1, LJ, Zh] without using Caffarelli-Gutiérrez's theory.

Theorem 3.4. *Assume $n = 2$ and $0 \leq \theta \leq \frac{1}{4}$. Let u be an entire smooth uniformly convex solution to* (3.1) *on $\mathbb{R}^2$. Then u is a quadratic polynomial.*

4. Interior Hölder estimates for linearized Monge-Ampère equations

The partial Legendre transform, though commonly applied to nonlinear elliptic equations, can also be effective for certain linear equations. In the context of Monge-Ampère equations, the linearized Monge-Ampère equation holds significant importance. It arises in several fundamental problems across various fields, including the Monge-Ampère type fourth order equations in the last section, the semigeostrophic equations in fluid mechanics, and the approximation of minimizers of convex functionals with convexity constraints in the calculus of variations, among others. For specific examples, see [Le, Section 1.2.1].

The linearized Monge-Ampère equation associated with a C^2 strictly convex potential ϕ, defined on a convex domain in $\mathbb{R}^n$, is expressed as

$$\sum_{i,j=1}^{n} \Phi^{ij} D_{ij} u \equiv \operatorname{trace}(\Phi D^2 u) = g,$$

where $\Phi = (\Phi^{ij})$ denotes the cofactor matrix of the Hessian of ϕ. The term "linearized" arises from linearizing the Monge-Ampère operator, as seen in

$$\det D^2(\phi + tu) = \det D^2\phi + t \operatorname{trace}(\Phi D^2 u) + \cdots + t^n \det D^2 u.$$

This shows that the equation is a type of linear second order equation, which can be degenerate or singular because its coefficient matrix is the cofactor matrix of the Hessian of a convex function. As a result, classical estimates for uniformly elliptic equations are not directly applicable.

To the best of the authors' knowledge, there are few references addressing the use of the partial Legendre transform for linearized Monge–Ampère equations. In an initial attempt in this direction, the first author [Wa] derives the Hölder estimates for solutions to linearized Monge-Ampère equations with possibly singular right-hand sides by applying the partial Legendre transform.

Consider the inhomogeneous linearized Monge-Ampère equation

$$\sum_{i,j=1}^{2} D_j \left(\Phi^{ij} D_i u\right) = \operatorname{div} F + f \tag{4.1}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^2$, where $\Phi = (\Phi^{ij})$ is the cofactor matrix of the Hessian matrix of a convex function $\phi \in C^2(\Omega)$, $F = (F^1, F^2) : \Omega \to \mathbb{R}^2$ is a vector field, and $f : \Omega \to \mathbb{R}$ is a function. We know that the associated functionals of (4.1) is

$$A(u) := \int_\Omega \phi_{x_2x_2}(u_{x_1})^2 - 2\phi_{x_1x_2}u_{x_1}u_{x_2} + \phi_{x_1x_1}(u_{x_2})^2 - 2F^1 u_{x_1} - 2F^2 u_{x_2} + 2fu \ \mathrm{d}x. \tag{4.2}$$

Since $\phi \in C^2(\Omega)$ is a convex function, we can perform the partial Legendre transform to ϕ. Denote $\phi^\star$ as the partial Legendre transform of ϕ, $\widetilde{u}(\xi, \eta) := u(\phi^\star_\xi, \eta)$, $\widetilde{F}(\xi, \eta) := F(\phi^\star_\xi, \eta)$ and $\widetilde{f}(\xi, \eta) := f(\phi^\star_\xi, \eta)$, then by (1.2) we know (4.2) becomes

$$A^\star(\widetilde{u}) := \int_{\Omega^\star} \left(-\frac{\phi^\star_{\eta\eta}}{\phi^\star_{\xi\xi}} \widetilde{u}_\xi^2 + \widetilde{u}_\eta^2 - 2\widetilde{F}^1 \widetilde{u}_\xi + 2\widetilde{F}^2 \phi^\star_{\xi\eta} \widetilde{u}_\xi - 2\widetilde{F}^2 \phi^\star_{\xi\xi} \widetilde{u}_\eta + 2\widetilde{f} \phi^\star_{\xi\xi} \widetilde{u} \right) \mathrm{d}\xi \mathrm{d}\eta.$$

Since u is a critical point of the functional $A(u)$, it follows that $\widetilde{u}$ is a critical point of the functional $A^\star(\widetilde{u})$, which means that $\widetilde{u}$ satisfies the Euler-Lagrange equation of $A^\star(\widetilde{u})$. Hence, we obtain:

Proposition 4.1 ([Wa, Proposition 2.1])**.** *Let u be a solution to* (4.1)*, then $\widetilde{u}$ satisfies*

$$\left(-\frac{\phi^\star_{\eta\eta}}{\phi^\star_{\xi\xi}} \widetilde{u}_\xi \right)_\xi + \widetilde{u}_{\eta\eta} = \left(\widetilde{F}^1 - \widetilde{F}^2 \phi^\star_{\xi\eta} \right)_\xi + \left(\widetilde{F}^2 \phi^\star_{\xi\xi} \right)_\eta + \widetilde{f} \phi^\star_{\xi\xi} \quad \text{in } \Omega^\star. \tag{4.3}$$

Similar to the approach for obtaining interior estimates for fourth order equations, we can derive interior estimates for linearized Monge-Ampère equations by estimating (4.3). We provide a sketch of the procedure here:

For any $x \in \Omega$, we denote $R = \frac{\mathrm{dist}(x,\partial\Omega)}{2}$. Without loss of generality, we assume $\mathcal{P}(x) = 0$. Assuming that ϕ satisfies $\lambda \le \det D^2\phi \le \Lambda$, hence we know that by [Liu, Lemma 2.1], there exists $\delta > 0$ depending on $C(R)$ such that $B_\delta(0) \subset \mathcal{P}(B_R(x))$. According to Proposition 4.1, we know that $\widetilde{u}$ satisfies (4.3) in $B_\delta(0)$ with

$$0 < \lambda \le \det D^2\phi = -\frac{\phi^\star_{\eta\eta}}{\phi^\star_{\xi\xi}} \le \Lambda.$$

This means that (4.3) is a uniformly elliptic equation in divergence form.

By the $W^{2,1+\varepsilon}$-estimate of Monge-Ampère equations [DFS, Sc], there exist $\varepsilon_0 > 0$ depending on λ, Λ, and $C_0 > 0$ depending on R, λ and Λ such that

$$\|D^2\phi\|_{L^{1+\varepsilon_0}(B_R(x))} \le C_0.$$

Hence, we have

$$\begin{aligned}\int_{B_\delta(0)} (\phi^\star_{\xi\xi})^{2+\varepsilon_0}\, \mathrm{d}\xi\mathrm{d}\eta &= \int_{\mathcal{P}^{-1}(B_\delta(0))} (\phi_{x_1x_1})^{-(2+\varepsilon_0)}\phi_{x_1x_1}\, \mathrm{d}x_1\mathrm{d}x_2\\ &= \int_{\mathcal{P}^{-1}(B_\delta(0))} (\phi_{x_1x_1})^{-(1+\varepsilon_0)}\, \mathrm{d}x_1\mathrm{d}x_2\\ &= \int_{\mathcal{P}^{-1}(B_\delta(0))} \left(\frac{\phi_{x_2x_2}}{\phi_{x_1x_1}\phi_{x_2x_2}}\right)^{1+\varepsilon_0} \mathrm{d}x_1\mathrm{d}x_2\\ &\le \lambda^{-(1+\varepsilon_0)} \int_{B_R(x)} (\phi_{x_2x_2})^{1+\varepsilon_0}\, \mathrm{d}x_1\mathrm{d}x_2\\ &\le C\lambda^{-(1+\varepsilon_0)}.\end{aligned}$$

Then

$$\int_{B_\delta(0)} (-\phi^\star_{\eta\eta})^{2+\varepsilon_0}\, \mathrm{d}\xi\mathrm{d}\eta \le \Lambda^{1+\varepsilon_0} \int_{B_\delta(0)} (\phi^\star_{\xi\xi})^{2+\varepsilon_0}\, \mathrm{d}\xi\mathrm{d}\eta \le C\Lambda^{1+\varepsilon_0}\lambda^{-(1+\varepsilon_0)}.$$

Finally, the standard $W^{2,p}$ theory of uniformly elliptic equations yields

$$\|\phi^\star_{\xi\eta}\|_{L^{2+\varepsilon_0}(B_\delta(0))} \le C.$$

With the assumptions on F and f, we know that the right-hand sides of (4.3) satisfy

$$\|\widetilde{F}^1 - \widetilde{F}^2\phi^\star_{\xi\eta}\|_{L^{2+\varepsilon_0}(B_\delta(0))} \le C,\ \ \|\widetilde{F}^2\phi^\star_{\xi\xi}\|_{L^{2+\varepsilon_0}(B_\delta(0))} \le C,$$

and

$$\begin{aligned}\int_{B_\delta(0)} \left|\widetilde{f}\phi^\star_{\xi\xi}\right|^{\frac{r(2+\varepsilon_0)}{1+\varepsilon_0+r}} \mathrm{d}\xi\mathrm{d}\eta &\le C \int_{\mathcal{P}^{-1}(B_\delta(0))} |f|^{\frac{r(2+\varepsilon_0)}{1+\varepsilon_0+r}} (\phi_{x_2x_2})^{\frac{(1+\varepsilon_0)(r-1)}{1+\varepsilon_0+r}}\, \mathrm{d}x_1\mathrm{d}x_2\\ &\le C\left(\int_{B_R(x)} |f|^r \mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{2+\varepsilon_0}{1+\varepsilon_0+r}} \left(\int_{B_R(x)} (\phi_{x_2x_2})^{1+\varepsilon_0}\, \mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{r-1}{1+\varepsilon_0+r}}\end{aligned}$$

$$\leq C\|f\|_{L^r(B_R(x))}^{\frac{r(2+\varepsilon_0)}{1+\varepsilon_0+r}}.$$

Note that $n = 2$ and $2+\varepsilon_0 > 2$, $\frac{r(2+\varepsilon_0)}{1+\varepsilon_0+r} > 1$ whenever $r > 1$, then the De Giorgi-Nash-Moser's theory [GT, Theorem 8.24] yields

$$\|\widetilde{u}\|_{C^\alpha(B_{\delta/2}(0))} \leq C\left(\|\widetilde{u}\|_{L^{\frac{p\varepsilon_0}{1+\varepsilon_0}}(B_\delta(0))} + k\right),$$

where

$$k = \|\widetilde{F}^1 - \widetilde{F}^2\phi^\star_{\xi\eta}\|_{L^{2+\varepsilon_0}(B_\delta(0))} + \|\widetilde{F}^2\phi^\star_{\xi\xi}\|_{L^{2+\varepsilon_0}(B_\delta(0))} + \|\widetilde{f}\phi^\star_{\xi\xi}\|_{L^{\frac{r(2+\varepsilon_0)}{1+\varepsilon_0+r}}(B_\delta(0))}.$$

Note that by Hölder's inequality there is

$$\begin{aligned}\left(\int_{B_\delta(0)} \widetilde{u}^{\frac{p\varepsilon_0}{1+\varepsilon_0}}\ \mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{1+\varepsilon_0}{p\varepsilon_0}} &= \left(\int_{\mathcal{P}^{-1}(B_\delta(0))} u^{\frac{p\varepsilon_0}{1+\varepsilon_0}}\phi_{x_1x_1}\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1+\varepsilon_0}{p\varepsilon_0}} \\ &\leq \|u\|_{L^p(B_R(x))}\cdot\|\phi_{x_1x_1}\|_{L^{1+\varepsilon_0}(B_R(x))}^{\frac{1+\varepsilon_0}{p\varepsilon_0}} \\ &\leq C\|u\|_{L^p(B_R(x))}.\end{aligned}$$

Hence, for the original function u, combining with the $C^{1,\alpha}$ estimate of Monge-Ampèpre equation we know that there exists a $\gamma \in (0,1)$ such that

$$\|u\|_{C^\gamma(\mathcal{P}^{-1}(B_{\delta/2}(0)))} \leq C\left(\|u\|_{L^p(B_R(x))} + \|F\|_{L^\infty(B_R(x))} + \|f\|_{L^r(B_R(x))}\right).$$

By a standard covering argument, we know that the estimate is true for any $\Omega' \subset\subset \Omega$, which is

Theorem 4.2 ([Wa, Theorem 1.1])**.** *Let $\phi \in C^2(\Omega)$ be a convex function satisfying $\lambda \leq \det D^2\phi \leq \Lambda$. Let $F := (F^1(x), F^2(x)) : \Omega \to \mathbb{R}^2$ be a bounded vector field and $f \in L^r(\Omega)$ for $r > 1$. Given $\Omega' \subset\subset \Omega$ and $p \in (0,+\infty)$, then for every solution u to* (4.1) *in Ω, there is*

$$\|u\|_{C^\gamma(\Omega')} \leq C\left(\|u\|_{L^p(\Omega)} + \|F\|_{L^\infty(\Omega)} + \|f\|_{L^r(\Omega)}\right),$$

where constant $\gamma > 0$ depending only on λ and Λ, and constant $C > 0$ depending only on p, r, λ, Λ, and $\mathrm{dist}(\Omega', \partial\Omega)$*.*

Using a similar argument, we can also establish a new Moser-Trudinger type inequality in two dimensions. To simplify the notation, we write

$$\|Du\|_\Phi^2 := \int_\Omega \Phi^{ij}D_iuD_ju\ \mathrm{d}x.$$

Theorem 4.3 ([Wa, Theorem 1.3])**.** *Let Ω be a uniformly convex domain in $\mathbb{R}^2$ and $\phi \in C^2(\Omega)$ be a convex function satisfying $\lambda \leq \det D^2\phi \leq \Lambda$. Assume that $\phi|_{\partial\Omega}$ and $\partial\Omega$ are of class C^3. For any $u \in C_0^\infty(\Omega)$, there exists a constant $C > 0$ depending* *only on λ, Λ, $\|\phi\|_{C^3(\partial\Omega)}$, the uniform convexity radius of $\partial\Omega$ and the C^3 regularity of $\partial\Omega$ such that*

$$\int_\Omega e^{\beta \frac{u^2}{\|Du\|_\Phi^2}}\ \mathrm{d}x_1\mathrm{d}x_2 \leq C|\Omega|^{\frac{\varepsilon_0}{2+\varepsilon_0}},$$

where $\beta \leq 4\pi\frac{1+\varepsilon_0}{2+\varepsilon_0}\min\{\lambda, 1\}$, and ε_0 depending only on λ and Λ is obtained by the global $W^{2,1+\varepsilon}$-estimate for Monge-Ampère equations.

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