2023 Summer School on Differential Geometry

Introduction to second order linear elliptic PDEs

Exam

Question 1. *Let u be a non-negative harmonic function in $B_R(0)$. Prove the Harnack inequality: for any $x \in B_R(0)$,*

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|}\, u(0) \leqslant u(x) \leqslant \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|}\, u(0).$$

Question 2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and L be a linear operator given by*

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c,$$

where $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$ and $a_{ij} = a_{ji}$. Suppose that L is strictly elliptic and there exists a function $v \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $v > 0$ in $\overline{\Omega}$ and $Lv \leqslant 0$ in Ω. Prove that if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \geqslant 0$ in Ω and $u \leqslant 0$ on $\partial\Omega$, then $u \leqslant 0$ in Ω.

Question 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and L be a linear operator given by*

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c,$$

where $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$, $a_{ij} = a_{ji}$ and $c \leqslant 0$. Suppose that L is strictly elliptic and Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$ (i.e. there exists a ball $B_R(y_0)$ such that $\Omega \cap B_R(y_0) = \emptyset$ and $\overline{\Omega} \cap \overline{B_R(y_0)} = \{x_0\}$). Prove that there exists a function $w_{x_0} \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $w_{x_0}(x_0) = 0$, $w_{x_0}(x) > 0$ for any $x \in \partial\Omega \setminus \{x_0\}$ and $Lw_{x_0} \leqslant -1$ in Ω.