A Two-Dimensional Anisotropic Stable Bernstein Theorem under an Ellipticity-Ratio Bound

Ling Wang, ling.wang@unibocconi.it
Department of Decision Sciences and BIDSA, Bocconi University, Milano, Italy

Abstract

These notes isolate an explicit two-dimensional stable Bernstein input for autonomous anisotropic surface energies. Let $F \in C^3(\mathbb{S}^2)$ be positive and uniformly elliptic, and set

$$\Psi_F := D^2_{\mathbb{S}^2} F + F\, g_{\mathbb{S}^2}.$$

Assume that

$$\lambda_F I \leq \Psi_F \leq \Lambda_F I, \qquad \kappa_F := \frac{\Lambda_F}{\lambda_F}.$$

We prove that if $\kappa_F < 8$, then every complete, two-sided, **F**-stationary and **F**-stable immersion $X : \Sigma^2 \to \mathbb{R}^3$ is an affine plane. The proof combines the anisotropic second variation formula, a two-dimensional algebraic identity, and Lin's sharp spectral threshold for equations of the form $\Delta u - \lambda K u = 0$. The constant 8 comes from the intrinsic coefficient $2/\kappa_F$ produced by stability and the threshold $\lambda > 1/4$.

The main purpose is to present, in a self-contained autonomous Euclidean form, the ellipticity-ratio calculation underlying the Bernstein input in the removable-singularity argument of De Rosa–Halavati–Wang. We also record the corresponding punctured formulation used after blow-up near an isolated singularity. In that setting the position-dependent integrand freezes to an autonomous integrand, and the blow-up limits form stable anisotropic minimal laminations in $\mathbb{R}^3 \setminus \{0\}$. The $\kappa < 8$ Bernstein input forces the leaves of these limiting laminations to be flat. This flatness is then converted into curvature decay, finite density, and the capacity argument used to remove the singularity.

1 Introduction

The Bernstein problem is one of the classical rigidity problems in minimal surface theory. In its graphical form, it asks whether an entire solution of the minimal surface equation over $\mathbb{R}^n$ must be affine. Equivalently, it asks whether an entire minimal graph

$$M = \{(x, u(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

must be a hyperplane. For the area functional, the answer is positive precisely in dimensions $n \leq 7$ and false in dimensions $n \geq 8$. This sharp picture is the result of a sequence of fundamental works: Bernstein's original theorem in dimension 2, the geometric-measure theoretic and regularity-theoretic developments of Fleming, De Giorgi, Almgren, and Simons, and the counterexamples of Bombieri–De Giorgi–Giusti in dimension 8 and above; see [Ber27, Fle62, DG65, Alm66, Sim68, BDGG69].

A natural extension of the classical problem is obtained by replacing the area functional by a parametric elliptic functional. In the hypersurface case, one considers a positive one-homogeneous integrand

$$\Phi : \mathbb{R}^{n+1} \setminus \{0\} \to (0, \infty)$$

and the anisotropic area functional

$$\Sigma \mapsto \int_\Sigma \Phi(\nu)\ \mathrm{d}\mu,$$

where ν is a chosen unit normal. The classical area functional corresponds to

$$\Phi(z) = |z|.$$

The associated Euler–Lagrange equation is the vanishing of anisotropic mean curvature. The anisotropic graphical Bernstein problem asks whether entire critical graphs for uniformly elliptic parametric integrands must be hyperplanes.

The anisotropic graphical problem has a different dimensional threshold from the isotropic one. For the area functional, entire minimal graphs are rigid up to dimension 7, and nonlinear entire examples exist in dimension 8 and above. For general uniformly elliptic parametric integrands, the sharp threshold is lower. The positive part was known classically in low dimensions: Jenkins [Jen61] proved the two-dimensional case for parametric variational problems, and Simon [Sim77] proved the three-dimensional case as part of his work on extensions of Bernstein's theorem. Thus arbitrary uniformly elliptic anisotropic integrands satisfy the graphical Bernstein property for graphs over $\mathbb{R}^n$ when $n \le 3$. The negative direction is more recent. Mooney [Moo22] constructed nonlinear entire solutions to equations of minimal surface type in dimension 6, for equations corresponding to parametric elliptic functionals. This gave counterexamples to the general anisotropic graphical Bernstein property in dimensions $n \ge 6$, leaving open the borderline dimensions 4 and 5. Mooney–Yang [MY24] completed the picture by constructing nonlinear entire anisotropic minimal graphs over $\mathbb{R}^4$. More precisely, they constructed a smooth nonlinear function $u : \mathbb{R}^4 \to \mathbb{R}$ and a uniformly elliptic integrand Φ on $\mathbb{R}^5$ such that the graph of u is a critical point of

$$A_\Phi(\Sigma) = \int_\Sigma \Phi(\nu)\ \mathrm{d}\mu.$$

Their construction gives a counterexample already over $\mathbb{R}^4$, and by product-type extensions gives counterexamples in all higher dimensions. Consequently, for arbitrary uniformly elliptic anisotropic integrands, the graphical Bernstein theorem is true exactly for $n \le 3$ and false for $n \ge 4$.

One should distinguish this general anisotropic threshold from the perturbative regime near the area functional. If the integrand is sufficiently close to

$$\Phi(z) = |z|$$

in a strong topology on the sphere, then the classical isotropic Bernstein theorem is stable under perturbation, and the rigidity range remains the same as for the area functional [Sim77]. Thus the Mooney–Yang examples necessarily use integrands that are quantitatively far from the area integrand. Their work also clarifies the relationship between anisotropic entire graphs and singular minimizing cones: the construction is inspired by foliations around anisotropic minimizing cones and produces examples with a range of possible growth rates. In this sense, the anisotropic graphical problem shows that uniform ellipticity alone is not enough to retain the dimension-7 Bernstein threshold of the area functional.

There is another Bernstein-type problem, logically distinct from the graphical one, concerning complete stable immersions. In the isotropic case, a two-sided minimal immersion

$$X : M^n \to \mathbb{R}^{n+1}$$

is stable if

$$\int_M |A|^2 u^2\,\mathrm{d}\mu \le \int_M |\nabla u|^2\,\mathrm{d}\mu \qquad \forall\, u \in C^1_c(M),$$

where A is the second fundamental form. The stable Bernstein problem asks whether every complete, two-sided, stable minimal immersion in Euclidean space must be a hyperplane. In dimension 2, namely for surfaces in $\mathbb{R}^3$, this was proved independently by do Carmo–Peng, Fischer-Colbrie–Schoen, and Pogorelov [dCP79, FCS80, Pog81]. Their theorem is one of the starting points of the modern theory of stable minimal hypersurfaces.

For stable minimal surfaces in $\mathbb{R}^3$, the proof is essentially spectral. If $M^2 \subset \mathbb{R}^3$ is minimal, then

$$|A|^2 = -2K,$$

where K is the Gauss curvature. Stability therefore gives

$$\int_M |\nabla u|^2 \, \mathrm{d}\mu \geq 2 \int_M |K| u^2 \, \mathrm{d}\mu.$$

This is much stronger than the sharp intrinsic spectral threshold

$$\int_M |\nabla u|^2 \, \mathrm{d}\mu \geq \mu \int_M |K| u^2 \, \mathrm{d}\mu, \qquad \mu > \frac{1}{4},$$

which forces $K \equiv 0$ on a complete surface with $K \leq 0$. The constant $1/4$ is sharp: on the hyperbolic plane $\mathbb{H}^2$, one has $K \equiv -1$ and

$$\lambda_0(-\Delta_{\mathbb{H}^2}) = \frac{1}{4}.$$

Thus $\mathbb{H}^2$ satisfies the endpoint inequality with $\mu = 1/4$, but is not flat. This spectral viewpoint is closely connected to the work of Fischer-Colbrie–Schoen, do Carmo–Peng, Gulliver–Lawson, Lin's eigenvalue theorem, and later refinements such as Castillon's theorem for Schrödinger operators on surfaces [dCP79, FCS80, GL86, Lin91, Cas06, McK70].

In higher dimensions, the stable Bernstein problem for the area functional is substantially more difficult. Classical curvature estimates and compactness theorems for stable minimal hypersurfaces were developed by Schoen–Simon–Yau and Schoen–Simon [SSY75, SS81]. These results imply, among other things, flatness under suitable volume growth hypotheses in the low-dimensional stable range. In particular, the strategy of proving Euclidean volume growth first and then applying stable curvature estimates has remained an important theme in later developments.

The first major breakthrough beyond the surface case was due to Chodosh–Li [CL24], who proved that every complete, two-sided, stable minimal hypersurface in $\mathbb{R}^4$ is flat. Their original proof uses the nonparabolicity of the stable hypersurface and a careful analysis of level sets of a positive Green's function. More precisely, if u is a positive Green's function on the hypersurface, one studies the geometry of the level sets

$$\Sigma_t = \{u = t\}$$

and derives monotonicity and integral estimates strong enough to force flatness. This proof is closely tied to scalar curvature ideas and to the macroscopic geometry of three-manifolds.

Chodosh–Li [CL23] later gave a second proof in their work on stable anisotropic minimal hypersurfaces in $\mathbb{R}^4$. Although that paper is anisotropic in scope, in the isotropic case it yields an alternative proof of the $\mathbb{R}^4$ stable Bernstein theorem. The method is closer to techniques from positive scalar curvature: one derives intrinsic cubic volume growth from stability, using a localization argument inspired by μ-bubbles and scalar curvature comparison. This second proof is especially relevant to anisotropic questions because it is designed to be robust under sufficiently small C^2-perturbations of the area integrand.

A further independent proof in $\mathbb{R}^4$ was given by Catino–Mastrolia–Roncoroni [CMR24]. Their argument uses a conformal deformation of the induced metric by a positive solution of the stability equation, together with weighted comparison and integral estimates. This approach is again inspired by the classical Fischer-Colbrie conformal method, but it packages the Chodosh–Li flatness theorem in a different geometric form. In particular, it gives a useful bridge between stability, conformal geometry, and weighted curvature comparison.

The next breakthrough was the theorem of Chodosh–Li–Minter–Stryker [CLMS26] in $\mathbb{R}^5$. They proved that every complete, two-sided, stable minimal hypersurface in $\mathbb{R}^5$ is flat. Their proof is based on the Gulliver–Lawson conformal metric

$$\widetilde{g} = r^{-2} g,$$

where r is the Euclidean distance to the origin, and on the observation that stability gives a form of spectral positivity for curvature of $(M, \widetilde{g})$. The relevant curvature quantity in dimension four is not merely scalar curvature but a spectral version of positive bi-Ricci curvature. This spectral positivity is then converted, through μ-bubble localization and volume comparison, into Euclidean volume growth, after which the classical curvature estimates imply flatness.

Mazet [Maz24] subsequently refined this strategy and proved the stable Bernstein theorem in $\mathbb{R}^6$. His proof follows the Chodosh–Li–Minter–Stryker framework but uses the volume estimate of Antonelli–Xu and a more delicate combination of spectral curvature inequalities. Thus the Euclidean stable Bernstein problem for complete two-sided stable minimal hypersurfaces is now known to have a positive answer through ambient dimension 6. The next unresolved Euclidean case is ambient dimension 7. In higher dimensions one encounters the classical obstructions coming from area-minimizing cones and their smoothings; in particular, singular area-minimizing cones appear in ambient dimension 8, while smooth nonflat entire minimal graphical examples exist from ambient dimension 9 onward [BDGG69, HS85].

There have also been recent expository and methodological refinements of these arguments. Antonelli–Xu [AX24] gave a note that reorganizes several recent proofs through a direct μ-bubble construction and clarifies the algebraic structure behind the dimension restrictions. Their perspective helps explain why the methods work in dimensions up to the known range and where the obstruction appears in the next borderline dimension. Altogether, these developments show that stable Bernstein-type rigidity in higher dimensions is highly sensitive to dimension and relies on converting the stability inequality into geometric positivity: scalar curvature in dimension 3, bi-Ricci or weighted spectral curvature in higher dimensions, and ultimately volume growth estimates that allow one to invoke the classical stable curvature theory.

The anisotropic stable Bernstein problem is more delicate. For elliptic parametric integrals, the first and second variation formulas and the corresponding regularity theory were developed in the work of Allard and White; see [All83, Whi87]. White proved curvature estimates and compactness theorems for surfaces stationary for parametric elliptic functionals in three-manifolds [Whi87]. Lin's papers [Lin90, Lin91] are especially relevant for the present note, but in a slightly different way. The spectral paper [Lin91] proves an eigenvalue nonexistence theorem for

$$\Delta_g u - \lambda K u = 0$$

on complete conformal metrics on the disk. In particular, under the additional assumption $K \leq 0$, Lin proves the sharp nonexistence threshold

$$\lambda > \frac{1}{4}.$$

Lin's work on elliptic parametric integrals [Lin90] applies this spectral perspective to stable surfaces with constant coefficients and obtains a Bernstein-type corollary under a $C^{2,\alpha}$-closeness assumption to the area integrand. This is a qualitative close-to-area assumption. One point of the present note is to separate from it the elementary ellipticity-ratio calculation that gives the explicit condition

$$\kappa_F < 8$$

in the autonomous two-dimensional case.

Further anisotropic stability results have appeared in several directions. Winklmann [Win05] proved pointwise curvature estimates for $\mathbf{F}$-stable hypersurfaces in dimensions $n \leq 5$, assuming that the integrand is sufficiently close to the area integrand. More recently, Chodosh–Li [CL23] studied stable anisotropic minimal hypersurfaces in $\mathbb{R}^4$, proving intrinsic cubic volume growth under a C^2-closeness assumption to the area functional and obtaining interior volume bounds with explicit constants. Li–Xia [LX25] proved stable anisotropic Bernstein theorems in $\mathbb{R}^5$ and $\mathbb{R}^6$, under a suitable C^4-closeness assumption to the area functional. Related anisotropic rigidity questions also arise in capillarity; for example, Guo–Xia [GX23] proved a Bernstein-type theorem for stable anisotropic capillary minimal surfaces in a three-dimensional half-space under a Euclidean area growth assumption. These works illustrate that stable anisotropic Bernstein theorems depend sensitively on dimension, stability, growth hypotheses, and quantitative closeness or ellipticity assumptions.

A further motivation comes from the recent work [DHW26], where De Rosa–Halavati–Wang develop an anisotropic analogue of the Meeks–Simon–Yau theory and an anisotropic Simon–Smith min–max theory with genus bounds. Their results show, among other things, that in every closed three-manifold endowed with an even elliptic anisotropic integrand, one can construct anisotropic min–max sequences within fixed isotopy classes whose limits are stable anisotropic minimal surfaces, smooth except possibly at a single point. They then prove two independent removable-singularity theorems for anisotropic minimal surfaces that are smooth and stable away from finitely many points. One criterion is based on an ellipticity-ratio bound; the other is based on a C^3-pinching condition.

The first of these removable-singularity criteria is the one most closely related to the present note. Let p be an isolated singular point. Since the integrand in [DHW26] is generally position-dependent, one freezes it at p, obtaining an autonomous integrand $F(p,\cdot)$ on $\mathbb{S}^2$. If $\Psi_F(p,\nu)$ denotes the ellipticity tensor of this frozen integrand, restricted to $T_\nu\mathbb{S}^2$, the relevant quantitative assumption is

$$\frac{\max_{\nu\in\mathbb{S}^2} \lambda_{\max}\Psi_F(p,\nu)}{\min_{\nu\in\mathbb{S}^2} \lambda_{\min}\Psi_F(p,\nu)} < 8.$$

Under this condition, the Bernstein input in [DHW26] says that a complete $F(p,\cdot)$-stationary and $F(p,\cdot)$-stable surface in $\mathbb{R}^3 \setminus \{0\}$, complete outside the origin in the natural punctured sense, is flat.

This Bernstein input is used in [DHW26] through a lamination argument. After rescaling near the singular point, the position-dependence of the integrand disappears to first order, and subsequences of the rescaled surfaces converge, away from the origin, to stable anisotropic minimal laminations for the frozen autonomous integrand. The Bernstein theorem is then applied leaf-by-leaf to the limiting lamination, forcing every leaf to be flat. This flatness of all blow-up laminations gives quantitative curvature decay near the puncture, in the form of small-scale $|x||A|$-control. The curvature decay is then used to control the topology of small distance level sets and to obtain a uniform finite density bound at the singular point. Once finite density is available, a standard capacity argument extends the stability inequality across the puncture, and the anisotropic curvature estimates imply that the singularity is removable.

The purpose of these notes is deliberately narrower and more elementary. We isolate the autonomous Euclidean calculation behind this removable-singularity input. Let

$$F \in C^3(\mathbb{S}^2), \qquad F > 0,$$

and set

$$\Psi_F := D^2_{\mathbb{S}^2} F + F\, g_{\mathbb{S}^2}.$$

Assume that

$$\lambda_F I \le \Psi_F \le \Lambda_F I$$

on $T\mathbb{S}^2$, and define the ellipticity ratio

$$\kappa_F := \frac{\Lambda_F}{\lambda_F}.$$

We prove that if $\kappa_F < 8$, then every complete, connected, two-sided, stable **F**-minimal immersion $X : \Sigma^2 \to \mathbb{R}^3$ is an affine plane. We also formulate the punctured version that is used, after blow-up, in the removable-singularity argument of [DHW26].

The proof is short, but the constant is worth recording explicitly. Along an **F**-stationary surface, the anisotropic mean curvature equation is

$$\mathrm{tr}_\Sigma(\Psi_F(\nu)S) = 0,$$

where S is the shape operator. In dimension two, this equation gives the exact pointwise identity

$$\mathrm{tr}_\Sigma(\Psi_F(\nu)S^2) = \mathrm{tr}_\Sigma(\Psi_F(\nu))\, |K|.$$

Consequently, the anisotropic stability inequality implies

$$2\lambda_F \int_\Sigma |K| u^2 \,\mathrm{d}\mu \le \Lambda_F \int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu,$$

or equivalently

$$\int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu \ge \frac{2}{\kappa_F} \int_\Sigma |K| u^2 \,\mathrm{d}\mu.$$

The intrinsic spectral Bernstein theorem applies precisely when

$$\frac{2}{\kappa_F} > \frac{1}{4},$$

which is equivalent to

$$\kappa_F < 8.$$

This is the origin of the number 8. We do not claim that $\kappa_F < 8$ is the optimal threshold for the anisotropic stable Bernstein problem. Rather, 8 is the explicit threshold produced by the classical two-dimensional spectral argument. The endpoint issue is intrinsic to this method: when

$$\mu = \frac{1}{4},$$

the hyperbolic plane satisfies the corresponding spectral inequality but has $K \equiv -1$. Thus the strict inequality $\kappa_F < 8$ is forced by the spectral route used here.

Thus the note has two complementary roles. First, it gives a self-contained proof of the autonomous two-dimensional Bernstein theorem under the explicit ratio bound $\kappa_F < 8$. Second, it explains how this theorem should be read inside the blow-up argument of [DHW26]: after rescaling near a puncture, the position-dependence of the integrand freezes to an autonomous integrand, the limit objects are stable for this frozen integrand, and the spectral Bernstein input forces the leaves of the limiting lamination to be flat.

The rest of the paper is organized as follows. Section 2 collects the preliminary material and the main statements: the anisotropic notation, the complete and punctured Bernstein theorems, Lin's spectral input, the anisotropic first and second variation formulas, and the elementary two-dimensional algebraic identity. Section 3 proves the complete Bernstein theorem. Section 4 records the punctured version and explains how it enters the lamination-based removable-singularity argument of [DHW26]. The final remarks discuss the constant 8, the endpoint obstruction, Lin's theorem, and the limitation of this autonomous argument for genuinely position-dependent integrands.

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2 Preliminaries and main statements

In this section we fix the anisotropic notation, state the two Bernstein inputs, recall the spectral theorem that will be used later, and record the first variation, second variation, and algebraic identities needed in the proof.

2.1 Anisotropic notation and the main statements

Let

$$F \in C^3(\mathbb{S}^2), \qquad F > 0.$$

Define the one-homogeneous extension of F to $\mathbb{R}^3 \setminus \{0\}$ by

$$\Phi_F(z) := |z| F\left(\frac{z}{|z|}\right).$$

For each $\nu \in \mathbb{S}^2$, define the anisotropic Hessian

$$\Psi_F(\nu) := D^2_{\mathbb{S}^2} F(\nu) + F(\nu) g_{\mathbb{S}^2}.$$

Equivalently,

$$\Psi_F(\nu) = D^2 \Phi_F(\nu)\big|_{\nu^\perp}.$$

We assume that F is uniformly elliptic: there exist constants

$$0 < \lambda_F \le \Lambda_F < \infty$$

such that

$$\lambda_F |\xi|^2 \le \Psi_F(\nu)(\xi, \xi) \le \Lambda_F |\xi|^2$$

for every $\nu \in \mathbb{S}^2$ and every

$$\xi \in T_\nu \mathbb{S}^2 \simeq \nu^\perp.$$

The anisotropic ellipticity ratio is

$$\kappa_F := \frac{\Lambda_F}{\lambda_F}.$$

Let

$$X : \Sigma^2 \to \mathbb{R}^3$$

be an oriented two-sided immersion with unit normal ν. We identify

$$T_p\Sigma = \nu(p)^\perp \simeq T_{\nu(p)}\mathbb{S}^2$$

when evaluating $\Psi_F(\nu)$ on tangent vectors. The anisotropic area functional is

$$\mathbf{F}(\Sigma) := \int_\Sigma F(\nu)\,\mathrm{d}\mu.$$

The first result is the complete autonomous Bernstein theorem.

Theorem 1 (Anisotropic stable Bernstein theorem under $\kappa_F < 8$)**.** *Let*

$$X : \Sigma^2 \to \mathbb{R}^3$$

be a complete, connected, oriented, two-sided smooth immersion without boundary. Assume that Σ is $\mathbf{F}$-stationary and $\mathbf{F}$-stable with respect to compactly supported normal variations. If $\kappa_F < 8$, then $X(\Sigma)$ is an affine plane.

We shall also use the corresponding punctured formulation. This is the form naturally arising after rescaling near an isolated point in the removable-singularity argument of [DHW26].

Theorem 2 ([DHW26, Theorem 5.2])**.** *Let $F \in C^3(\mathbb{S}^2)$ be autonomous and uniformly elliptic, and assume $\kappa_F < 8$. Let $X : \Sigma^2 \to \mathbb{R}^3 \setminus \{0\}$ be a connected, oriented, two-sided smooth immersion without boundary which is $\mathbf{F}$-stationary and $\mathbf{F}$-stable with respect to compactly supported normal variations. Assume moreover that Σ is complete with respect to the conformally rescaled metric $\widehat{g} := |X|^{-2}g$, where g is the induced metric. Then $X(\Sigma)$ is contained in an affine plane. In particular, the closure of $X(\Sigma)$ in $\mathbb{R}^3$ is a planar surface.*

2.2 Spectral input

We use the sign convention

$$\Delta = \operatorname{div}\nabla,$$

so that

$$\int_\Sigma |\nabla\varphi|^2\,\mathrm{d}\mu = -\int_\Sigma \varphi\,\Delta\varphi\,\mathrm{d}\mu$$

for compactly supported φ.

The spectral ingredient used below is the following theorem of Lin. Lin states the result for complete conformal metrics on the unit disk.

Theorem 3 ([Lin91, Theorem A])**.** *Let*

$$g = \rho^2|\,\mathrm{d}z|^2$$

be a complete conformal metric on the unit disk $D \subset \mathbb{C}$, and let K be its Gauss curvature. If $K \le 0$, then, for every

$$\lambda > \frac{1}{4},$$

the equation

$$\Delta_g u - \lambda K u = 0 \tag{1}$$

has no positive solution on D.

Remark 1. *Lin also proves a weaker threshold $\lambda > 1/2$ without the assumption $K \leq 0$. The improvement from $1/2$ to $1/4$ is exactly the case relevant here, because the anisotropic algebraic identity below gives $K \leq 0$ for two-dimensional autonomous stationary surfaces.*

We shall combine Lin's theorem with the following standard Allegretto–Piepenbrink, or Fischer-Colbrie–Schoen, ground-state principle for Schrödinger operators [FCS80].

Lemma 1 (Ground-state principle)**.** *Let (M, g) be a complete Riemannian surface and let $V \in L^\infty_{\mathrm{loc}}(M)$. If*

$$\int_M \left(|\nabla\varphi|^2 + V\varphi^2\right) \mathrm{d}\mu \geq 0 \qquad \forall\varphi \in C_c^\infty(M),$$

then there exists a positive C^2 solution w of

$$\Delta w - V w = 0$$

on M.

The following intrinsic consequence is the spectral Bernstein theorem used in the proof of Theorem 1.

Theorem 4 (Intrinsic spectral Bernstein theorem)**.** *Let (Σ^2, g) be a complete surface with Gauss curvature $K \leq 0$. Assume that there exists a constant*

$$\mu > \frac{1}{4}$$

such that

$$\int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu \geq \mu \int_\Sigma |K| u^2 \,\mathrm{d}\mu \tag{2}$$

for every $u \in C_c^1(\Sigma)$. Then $K \equiv 0$.

Proof. First suppose that Σ is compact. Taking $u \equiv 1$ in (2) gives

$$0 \geq \mu \int_\Sigma |K| \,\mathrm{d}\mu.$$

Since $\mu > 0$, it follows that $K \equiv 0$.

We may therefore assume that Σ is noncompact. Since $K \leq 0$, we have $|K| = -K$. Hence (2) is equivalent to

$$\int_\Sigma \left(|\nabla u|^2 + \mu K u^2\right) \mathrm{d}\mu \geq 0 \qquad \forall u \in C_c^1(\Sigma). \tag{3}$$

In particular, the same inequality holds for all $u \in C_c^\infty(\Sigma)$. Applying Lemma 1 with $V = \mu K$, we obtain a positive solution w of

$$\Delta w - \mu K w = 0 \tag{4}$$

on Σ.

Let

$$\pi : \widetilde{\Sigma} \to \Sigma$$

be the universal cover, equipped with the lifted metric. The lifted function

$$\widetilde{w} = w \circ \pi$$

is positive and satisfies

$$\Delta_{\widetilde{g}} \widetilde{w} - \mu \widetilde{K} \widetilde{w} = 0 \tag{5}$$

on $\widetilde{\Sigma}$. Since Σ is noncompact, its universal cover is not conformally equivalent to $\mathbb{S}^2$. By uniformization, $\widetilde{\Sigma}$ is conformally equivalent either to $\mathbb{C}$ or to the unit disk D.

If $\widetilde{\Sigma}$ is conformally equivalent to D, then (5) contradicts Lin's theorem 3, because $\mu > 1/4$ and $\widetilde{K} \leq 0$. Therefore this case cannot occur.

It remains that $\widetilde{\Sigma}$ is conformally equivalent to $\mathbb{C}$. Since $\widetilde{K} \leq 0$, equation (5) gives

$$\Delta_{\widetilde{g}} \widetilde{w} = \mu \widetilde{K} \widetilde{w} \leq 0.$$

In a global conformal coordinate on $\mathbb{C}$, this says that $\widetilde{w}$ is a positive superharmonic function on the complex plane. Hence $\widetilde{w}$ is constant. Returning to (5), we obtain

$$\mu \widetilde{K} \widetilde{w} = 0.$$

Since $\mu > 0$ and $\widetilde{w} > 0$, it follows that $\widetilde{K} \equiv 0$, and therefore $K \equiv 0$ on Σ. $\square$

Remark 2 (Endpoint failure)**.** *The conclusion of Theorem 4 is false at the endpoint*

$$\mu = \frac{1}{4}.$$

Indeed, on $\mathbb{H}^2$, one has $K \equiv -1$ and

$$\lambda_0(-\Delta_{\mathbb{H}^2}) = \frac{1}{4};$$

see, for instance, McKean [McK70]. Consequently,

$$\int_{\mathbb{H}^2} |\nabla u|^2 \, d\mu_{\mathbb{H}} \geq \frac{1}{4} \int_{\mathbb{H}^2} |K| u^2 \, d\mu_{\mathbb{H}} \qquad \forall u \in C_c^1(\mathbb{H}^2),$$

but $K \not\equiv 0$. Thus the strict inequality

$$\mu > \frac{1}{4}$$

is essential.

2.3 Anisotropic first and second variations

Let

$$X : \Sigma^2 \to \mathbb{R}^3$$

be an oriented immersion with unit normal ν, and let

$$S : T\Sigma \to T\Sigma$$

be the shape operator. We use a fixed sign convention for S. Changing the convention replaces S by $-S$. The first variation formula therefore changes by an overall sign, but the Euler–Lagrange equation below is unchanged. The second variation formula only involves S^2, and is therefore unaffected by this choice.

The first variation formula for the anisotropic area functional gives

$$\delta\mathbf{F}(u\nu) = -\int_\Sigma u \,\mathrm{tr}_\Sigma\big(\Psi_F(\nu)S\big)\,\mathrm{d}\mu.$$

Thus Σ is $\mathbf{F}$-stationary if and only if

$$\mathrm{tr}_\Sigma\big(\Psi_F(\nu)S\big) = 0. \tag{6}$$

This is the vanishing anisotropic mean curvature equation.

The second variation formula at an $\mathbf{F}$-stationary immersion gives, for every $u \in C_c^1(\Sigma)$,

$$\delta^2\mathbf{F}(u\nu, u\nu) = \int_\Sigma \left[\Psi_F(\nu)(\nabla u, \nabla u) - \mathrm{tr}_\Sigma\big(\Psi_F(\nu)S^2\big)u^2\right]\,\mathrm{d}\mu.$$

Therefore $\mathbf{F}$-stability is equivalent to

$$\int_\Sigma \mathrm{tr}_\Sigma\big(\Psi_F(\nu)S^2\big)u^2\,\mathrm{d}\mu \le \int_\Sigma \Psi_F(\nu)(\nabla u, \nabla u)\,\mathrm{d}\mu \tag{7}$$

for every $u \in C_c^1(\Sigma)$. These formulas are standard in the theory of autonomous elliptic parametric integrals; see [All83, Whi87]. Lin's work uses the associated stability equation on the universal cover in the constant-coefficient surface case [Lin90, Lin91]. They are the anisotropic analogues of the first variation equation $H = 0$ and the stability inequality

$$\int_\Sigma |A|^2u^2\,\mathrm{d}\mu \le \int_\Sigma |\nabla u|^2\,\mathrm{d}\mu$$

for stable minimal surfaces in $\mathbb{R}^3$.

2.4 The two-dimensional algebraic identity

The key point in dimension two is that the anisotropic stationarity equation forces a precise relationship between the anisotropic Jacobi potential and the Gauss curvature.

Lemma 2. *Let $p \in \Sigma$. Suppose*

$$\mathrm{tr}_\Sigma(\Psi_F(\nu)S) = 0$$

at p. Then

$$K(p) \le 0$$

and

$$\mathrm{tr}_\Sigma(\Psi_F(\nu)S^2) = \mathrm{tr}_\Sigma(\Psi_F(\nu))\,|K|.$$

In particular,

$$\mathrm{tr}_\Sigma(\Psi_F(\nu)S^2) \ge 2\lambda_F|K|.$$

Proof. Fix $p \in \Sigma$. Choose an orthonormal basis of $T_p\Sigma$ diagonalizing $\Psi_F(\nu)$. In this basis,

$$\Psi_F(\nu) = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad \lambda_F \le b_i \le \Lambda_F.$$

Write

$$S = \begin{pmatrix} a & c \\ c & d \end{pmatrix}.$$

The stationarity equation becomes

$$b_1 a + b_2 d = 0.$$

Hence

$$d = -\frac{b_1}{b_2} a.$$

Therefore

$$K = \det S = ad - c^2 = -\frac{b_1}{b_2} a^2 - c^2 \le 0.$$

Thus

$$|K| = \frac{b_1}{b_2} a^2 + c^2. \tag{8}$$

On the other hand,

$$\operatorname{tr}(\Psi_F S^2) = b_1(a^2 + c^2) + b_2(c^2 + d^2).$$

Using

$$d = -\frac{b_1}{b_2} a,$$

we obtain

$$\operatorname{tr}(\Psi_F S^2) = (b_1 + b_2)\left(\frac{b_1}{b_2} a^2 + c^2\right).$$

By (8),

$$\operatorname{tr}(\Psi_F S^2) = \operatorname{tr}(\Psi_F)|K|. \tag{9}$$

Since

$$\operatorname{tr}(\Psi_F) = b_1 + b_2 \ge 2\lambda_F,$$

we get

$$\operatorname{tr}(\Psi_F S^2) \ge 2\lambda_F |K|.$$

This proves the lemma. □

3 Proof of the main theorem

Proof of Theorem 1. Let $u \in C_c^1(\Sigma)$. By stability,

$$\int_\Sigma \operatorname{tr}_\Sigma\big(\Psi_F(\nu) S^2\big) u^2 \, \mathrm{d}\mu \le \int_\Sigma \Psi_F(\nu)(\nabla u, \nabla u) \, \mathrm{d}\mu.$$

By Lemma 2,

$$\operatorname{tr}_\Sigma\big(\Psi_F(\nu) S^2\big) \ge 2\lambda_F |K|.$$

By ellipticity,

$$\Psi_F(\nu)(\nabla u, \nabla u) \le \Lambda_F |\nabla u|^2.$$

Combining the last three inequalities gives

$$2\lambda_F \int_\Sigma |K| u^2 \, \mathrm{d}\mu \le \Lambda_F \int_\Sigma |\nabla u|^2 \, \mathrm{d}\mu.$$

Equivalently,

$$\int_\Sigma |\nabla u|^2 \, \mathrm{d}\mu \ge \frac{2}{\kappa_F} \int_\Sigma |K| u^2 \, \mathrm{d}\mu \tag{10}$$

for every $u \in C_c^1(\Sigma)$.

Moreover, Lemma 2 gives $K \le 0$ everywhere on Σ. Since $\kappa_F < 8$, we have

$$\frac{2}{\kappa_F} > \frac{1}{4}.$$

Therefore Theorem 4 applies to (10), and we conclude that $K \equiv 0$.

By the algebraic identity (9),

$$\operatorname{tr}(\Psi_F S^2) = \operatorname{tr}(\Psi_F)|K| \equiv 0.$$

Since Ψ_F is positive definite and S is symmetric,

$$\operatorname{tr}(\Psi_F S^2) = \operatorname{tr}(S\Psi_F S) = \operatorname{tr}\big((\Psi_F^{1/2} S)^T(\Psi_F^{1/2} S)\big) \ge 0,$$

with equality only if $S \equiv 0$.. Thus the second fundamental form vanishes identically. Hence the immersion is totally geodesic.

Since Σ is connected, the unit normal ν is constant, and therefore $X(\Sigma) \subset P$ for some affine plane $P \subset \mathbb{R}^3$. Moreover, $X : \Sigma \to P$ is a local isometry for the induced metric. Since Σ is complete and P is connected and complete, the standard lifting argument for geodesics shows that $X(\Sigma) = P$. Thus $X(\Sigma)$ is an affine plane. □

4 The punctured version and the lamination application

We now explain how the autonomous calculation above appears in the punctured setting used in [DHW26]. This section is not needed for the proof of Theorem 1; it is included to clarify the precise role of the Bernstein input in the removable-singularity argument. Throughout this section, when $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$, completeness outside the origin means completeness with respect to the conformally rescaled metric

$$\widehat{g} := r^{-2} g, \qquad r = |x|,$$

where g is the metric induced by the Euclidean metric on Σ.

4.1 The punctured Bernstein input

Let F be autonomous, and let $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$ be a connected two-sided immersed surface which is $\mathbf{F}$-stationary and $\mathbf{F}$-stable away from the origin. In this subsection we write A_Σ for the shape operator. The stability inequality is initially available for test functions $\varphi \in C_c^1(\Sigma)$, where Σ is regarded as a surface without the origin. Equivalently, these are compactly supported test functions away from the puncture in $\mathbb{R}^3$.

The algebraic part of the complete argument is unchanged. Stationarity gives

$$\operatorname{tr}_\Sigma(\Psi_F A_\Sigma) = 0.$$

The Cayley–Hamilton identity for the two-dimensional endomorphism A_Σ is

$$A_\Sigma^2 - H_\Sigma A_\Sigma + K_\Sigma I = 0.$$

After composing with Ψ_F and taking the trace, we obtain

$$\operatorname{tr}_\Sigma(\Psi_F A_\Sigma^2) - H_\Sigma \operatorname{tr}_\Sigma(\Psi_F A_\Sigma) + K_\Sigma \operatorname{tr}_\Sigma(\Psi_F) = 0.$$

Using the stationarity equation, this becomes

$$\operatorname{tr}_\Sigma(\Psi_F A_\Sigma^2) = -\operatorname{tr}_\Sigma(\Psi_F) K_\Sigma.$$

Moreover, as in Lemma 2, the stationarity equation and the positive definiteness of Ψ_F imply $K_\Sigma \le 0$. Hence

$$\operatorname{tr}_\Sigma(\Psi_F A_\Sigma^2) = \operatorname{tr}_\Sigma(\Psi_F)|K_\Sigma|.$$

Define

$$m_F := \min_{\nu\in\mathbb{S}^2} \lambda_{\min}\Psi_F(\nu), \qquad M_F := \max_{\nu\in\mathbb{S}^2} \lambda_{\max}\Psi_F(\nu).$$

Then $m_F > 0$, and the ellipticity-ratio assumption is

$$\frac{M_F}{m_F} < 8.$$

The stability inequality gives

$$\int_\Sigma \left[\Psi_F(\nabla\varphi, \nabla\varphi) - \operatorname{tr}_\Sigma(\Psi_F A_\Sigma^2)\varphi^2\right] \mathrm{d}\mu \ge 0.$$

Using

$$\Psi_F(\nabla\varphi, \nabla\varphi) \le M_F|\nabla\varphi|^2, \qquad \operatorname{tr}_\Sigma(\Psi_F) \ge 2m_F,$$

and $K_\Sigma \le 0$, we obtain

$$\int_\Sigma |\nabla\varphi|^2 \,\mathrm{d}\mu \ge \frac{2m_F}{M_F}\int_\Sigma |K_\Sigma|\varphi^2 \,\mathrm{d}\mu \qquad \forall \varphi \in C_c^1(\Sigma). \tag{11}$$

Equivalently,

$$\int_\Sigma \left(|\nabla\varphi|^2 + c_0 K_\Sigma \varphi^2\right) \mathrm{d}\mu \ge 0, \qquad c_0 := \frac{2m_F}{M_F}.$$

The condition $M_F/m_F < 8$ is precisely

$$c_0 > \frac{1}{4}.$$

The remaining point is conformal. The metric

$$\widehat{g} = r^{-2} g$$

is induced by the ambient cylindrical metric

$$r^{-2}\langle\cdot,\cdot\rangle$$

on $\mathbb{R}^3 \setminus \{0\}$, which is isometric to $\mathbb{S}^2 \times \mathbb{R}$. Thus completeness outside the origin gives ordinary completeness of $(\Sigma, \widehat{g})$. The punctured version of Lin's argument, in the form used in [DHW26, Theorem 5.2], then applies to the quadratic-form inequality (11): since $c_0 > 1/4$ and $K_\Sigma \le 0$, the conformal type is parabolic, namely $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$.

By the ground-state principle applied to the nonnegative quadratic form, there exists a positive solution of

$$-\Delta_\Sigma u + c_0 K_\Sigma u = 0.$$

Since $K_\Sigma \le 0$, this equation gives

$$\Delta_\Sigma u = c_0 K_\Sigma u \le 0.$$

Thus u is a positive superharmonic function on a parabolic Riemann surface, and therefore u is constant. It follows that

$$c_0 K_\Sigma u = 0.$$

Since $c_0 > 0$ and $u > 0$, we conclude that $K_\Sigma \equiv 0$. The algebraic identity then gives

$$\mathrm{tr}_\Sigma(\Psi_F A_\Sigma^2) \equiv 0.$$

Since Ψ_F is positive definite, this forces $A_\Sigma \equiv 0$. Therefore Σ is locally contained in affine planes. Since Σ is connected, $X(\Sigma)$ is contained in a single affine plane. Completeness outside the origin rules out deleting any additional interior set in that plane. Hence the closure of $X(\Sigma)$ in $\mathbb{R}^3$ is an affine plane. This proves Theorem 2 in the form needed for blow-up arguments.

4.2 From flat blow-up leaves to removability

We next spell out the lamination mechanism in [DHW26]. Let $F = F(x, \nu)$ be a smooth uniformly elliptic integrand in a ball $B_1 \subset \mathbb{R}^3$, and let $\Sigma \subset B_1 \setminus \{0\}$ be a properly embedded surface which is **F**-stationary and **F**-stable away from the origin. Assume that the frozen integrand $F(0, \cdot)$ satisfies

$$\frac{\max_{\nu \in \mathbb{S}^2} \lambda_{\max} \Psi_F(0, \nu)}{\min_{\nu \in \mathbb{S}^2} \lambda_{\min} \Psi_F(0, \nu)} < 8. \tag{12}$$

Here $\Psi_F(0, \nu)$ is the anisotropic Hessian of the frozen integrand $F(0, \cdot)$, restricted to $T_\nu \mathbb{S}^2$.

The first input is the local curvature estimate for stable anisotropic surfaces. Since Σ is stable in balls whose radius is comparable to the distance from the origin, one obtains the scale-invariant bound

$$|A_\Sigma(x)| \le \frac{C}{|x|}$$

near the puncture. Let $\rho_i \downarrow 0$, and consider the rescaled surfaces

$$\Sigma_i := \rho_i^{-1} \Sigma.$$

The corresponding rescaled integrands converge, on compact subsets of $\mathbb{R}^3 \setminus \{0\}$, to the frozen autonomous integrand $F(0, \cdot)$. The curvature estimate gives locally bounded curvature for Σ_i on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Hence, after passing to a subsequence, the Σ_i converge as laminations to a limiting lamination

$$\mathcal{L} \subset \mathbb{R}^3 \setminus \{0\}.$$

Each leaf of $\mathcal{L}$ is $F(0, \cdot)$-stationary and $F(0, \cdot)$-stable.

By Theorem 2, every leaf of $\mathcal{L}$ is planar. Since the leaves are disjoint, the limiting lamination is a flat lamination. This flatness holds for every blow-up sequence. Consequently, there can be no nonzero curvature concentration at the puncture. Indeed, if there were an $\varepsilon_0 > 0$ and a sequence $x_i \in \Sigma$, $x_i \to 0$, such that

$$|x_i| \, |A_\Sigma(x_i)| \ge \varepsilon_0,$$

then rescaling by $|x_i|^{-1}$ would produce a blow-up lamination with a leaf having nonzero curvature at a point a definite distance from the origin, contradicting the flatness of all leaves. Therefore

$$\lim_{x \to 0,\, x \in \Sigma} |x| \, |A_\Sigma(x)| = 0. \tag{13}$$

This curvature decay is the key geometric output of the Bernstein step. It is then used to recover the density control that, in the isotropic case, would usually follow from monotonicity. More precisely, once (13) is known, the squared distance function

$$f(x) = |x|^2\big|_\Sigma$$

has no saddle-type critical points on sufficiently small annuli around the origin. This controls the topology of the level sets $\Sigma \cap \partial B_r$ for small r. Combined with Gauss–Bonnet and the smallness of $|x|\,|A_\Sigma|$, this gives a uniform finite-density bound,

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^2(\Sigma \cap B_r)}{r^2} < \infty.$$

Finite density implies that the puncture has zero $W^{1,2}$-capacity on Σ. A logarithmic cutoff argument then extends the stationarity and stability inequalities from test functions supported away from the origin to test functions whose supports cross the origin. Finally, the two-dimensional anisotropic curvature estimates and regularity theory imply that the closure $\overline{\Sigma} \cap B_1$ is a smooth embedded **F**-stationary surface. This is the Bernstein-based removable-singularity criterion in [DHW26, Theorem 5.1].

The important point is conceptual. The ratio condition (12) is not used directly to estimate the original position-dependent equation. It is used after blow-up, where the position-dependence of the integrand freezes to the autonomous integrand $F(0,\cdot)$. The autonomous stable Bernstein theorem then acts leaf-by-leaf on the limiting lamination. Flatness of the blow-up leaves is converted into curvature decay and density control for the original punctured surface, and only after this step does the standard capacity-removability argument become available.

5 Further comments

We finish with a few comments on the constant, the relation with Lin's theorem, and the genuinely autonomous nature of the argument.

Remark 3 (Origin of the constant 8)**.** *The constant* 8 *comes from the chain*

$$\mathbf{F}\text{-}\textit{stability} \Longrightarrow \int_\Sigma |\nabla u|^2 \,\mathrm{d}\mu \geq \frac{2}{\kappa_F} \int_\Sigma |K| u^2 \,\mathrm{d}\mu,$$

together with the sharp intrinsic spectral threshold

$$\mu > \frac{1}{4}.$$

Thus

$$\frac{2}{\kappa_F} > \frac{1}{4} \quad \Longleftrightarrow \quad \kappa_F < 8.$$

The essential two-dimensional identity is

$$\mathrm{tr}(\Psi_F S^2) = \mathrm{tr}(\Psi_F)|K|.$$

If one compares the anisotropic Jacobi potential only with $|A|^2$*, one obtains a weaker pinching condition. The point of the present calculation is that the stationarity equation gives an exact identity, not merely a rough comparison.*

Remark 4 (Endpoint issue)**.** *The endpoint $\kappa_F = 8$ corresponds to the intrinsic spectral endpoint*

$$\frac{2}{\kappa_F} = \frac{1}{4}.$$

The intrinsic spectral Bernstein theorem is false at this endpoint, as shown by $\mathbb{H}^2$, where $K \equiv -1$ and

$$\lambda_0(-\Delta_{\mathbb{H}^2}) = \frac{1}{4}.$$

Therefore the present scalar spectral method cannot treat $\kappa_F = 8$. This does not prove that $\kappa_F = 8$ is the sharp threshold for the anisotropic stable Bernstein problem. It only shows that strict pinching is necessary for the particular route used here.

Remark 5 (Relation with Lin's theorem)**.** *The central spectral input is Lin's nonexistence theorem [Lin91] for positive solutions of*

$$\Delta_g u - \lambda K u = 0$$

on complete conformal metrics on the disk, with the sharp threshold

$$\lambda > \frac{1}{4}$$

under the assumption $K \leq 0$. Lin's estimates for elliptic parametric integrals [Lin90] apply this spectral perspective to stable surfaces with constant coefficients and yield a plane theorem under a $C^{2,\alpha}$-closeness assumption to the area integrand.

The present note extracts a ratio version of the same spectral route. In the autonomous two-dimensional case, the anisotropic stationarity equation gives

$$\mathrm{tr}(\Psi_F S^2) = \mathrm{tr}(\Psi_F)|K|,$$

and stability therefore implies the intrinsic spectral inequality with coefficient $2/\kappa_F$. Combining this coefficient with Lin's $1/4$-threshold yields the explicit sufficient condition $\kappa_F < 8$. In [DHW26], this explicit ratio condition is used for the frozen integrand in the blow-up analysis of isolated singularities.

Remark 6 (Relation with the min–max application)**.** *The theorem isolated here is the autonomous model for the Bernstein input used in [DHW26]. In that work the integrand is defined on the unit tangent bundle of a three-manifold and may depend on the base point. Near an isolated singular point p, blow-up freezes the integrand to $F(p, \cdot)$. The pointwise ellipticity condition used there is precisely a version of $\kappa_{F(p,\cdot)} < 8$ for this frozen integrand.*

The blow-up limit is a stable anisotropic lamination in $\mathbb{R}^3 \setminus \{0\}$, and each leaf satisfies the hypotheses of the punctured Bernstein input. Therefore every leaf is flat. This flat lamination conclusion is then converted into curvature decay, finite density, zero capacity of the puncture, and finally smooth removability. Thus the autonomous theorem proved here enters the min–max theory only after passing to blow-up limits and freezing the position-dependent integrand.

Remark 7 (Autonomous versus position-dependent integrands)**.** *The assumption that F depends only on the normal variable is essential for the argument. If $F = F(x, \nu)$, then the first variation contains additional terms involving derivatives of F with respect to the base point. Schematically, the stationarity equation takes the form*

$$\mathrm{tr}_\Sigma(\Psi_F(x,\nu)S) + \mathcal{E}_F(x, \nu, T\Sigma) = 0,$$

where $\mathcal{E}_F$ involves spatial derivatives such as $D_x F$ and mixed derivatives such as $D_x D_\nu F$. Thus one no longer has

$$\mathrm{tr}_\Sigma(\Psi_F(x,\nu)S) = 0.$$

In dimension two, the Cayley–Hamilton identity gives

$$S^2 - HS + KI = 0.$$

After multiplying by Ψ_F and taking the trace, one obtains

$$\mathrm{tr}(\Psi_F S^2) = -\,\mathrm{tr}(\Psi_F)K + H\ \mathrm{tr}(\Psi_F S).$$

Along a position-dependent stationary surface this becomes, schematically,

$$\mathrm{tr}(\Psi_F S^2) = -\,\mathrm{tr}(\Psi_F)K - H\,\mathcal{E}_F.$$

Consequently K need not be nonpositive, and the identity

$$\mathrm{tr}(\Psi_F S^2) = \mathrm{tr}(\Psi_F)|K|$$

is no longer available.

Moreover, the second variation for a position-dependent integrand also contains lower-order terms involving spatial and mixed derivatives of F. Hence stability does not directly imply the intrinsic inequality

$$\int_\Sigma |\nabla u|^2\ \mathrm{d}\mu \geq \frac{2}{\kappa_F}\int_\Sigma |K|u^2\ \mathrm{d}\mu.$$

The proof of Theorem 1 is therefore genuinely autonomous. For general position-dependent elliptic parametric integrals, an ellipticity ratio alone cannot control the additional spatial derivative terms.

Remark 8 (Open problem)**.** *It remains open whether the ellipticity-ratio assumption in Theorem 1 can be removed. Lin's eigenvalue theorem gives the sharp scalar threshold $1/4$, so the scalar comparison argument used here cannot go beyond*

$$\kappa_F < 8.$$

A proof without this restriction would have to use additional structure of the anisotropic problem, such as the Cahn–Hoffman field

$$\xi_F = D_{\mathbb{S}^2}F + F\nu,$$

the special tensor identity

$$\Psi_F = D^2_{\mathbb{S}^2}F + Fg_{\mathbb{S}^2},$$

or extrinsic information coming from the fact that the nonpositively curved metric is induced by a complete immersion in $\mathbb{R}^3$. In particular, any improvement beyond $\kappa_F < 8$ would have to use more than the intrinsic inequality

$$\int_\Sigma |\nabla u|^2\,\mathrm{d}\mu \geq \mu\int_\Sigma |K|u^2\,\mathrm{d}\mu.$$

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