ASYMPTOTIC EXPANSIONS OF THE FUNDAMENTAL SOLUTION FOR A MIXED LOCAL–NONLOCAL OPERATOR

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Abstract. In this paper, we study the fundamental solution of the mixed local–nonlocal operator $-\Delta + (-\Delta)^s$, $0 < s < 1$. We derive precise asymptotic expansions of the fundamental solution and its gradient, up to the first nontrivial correction term, both near the origin and at infinity. The expansions describe the transition between the local and nonlocal diffusion scales and include the critical regimes in which logarithmic terms occur. As applications of these asymptotics, we establish a distributional Bôcher-type theorem for nonnegative supersolutions with an isolated singularity. We further obtain quantitative positive and antisymmetric maximum principles on punctured balls.

1. Introduction

Mixed local–nonlocal operators arise when diffusion mechanisms of different orders act simultaneously. In this paper, we consider the operator

$$\mathscr{L}_s := -\Delta + (-\Delta)^s, \qquad 0 < s < 1.$$

At the probabilistic level, $-\mathscr{L}_s$ is the generator of the sum of a Brownian motion and an independent symmetric $2s$-stable process. This connects the operator with the potential theory of stable processes and mixed Brownian–stable processes. We refer to [13, 43, 16, 15] for representative results in this direction. Local diffusion coupled with jump processes also appears naturally in stochastic control and jump-diffusion problems [11, 12, 25], as well as in population dynamics and ecological dispersal models [24, 22].

From the PDE point of view, equations containing both differential and integral terms belong to the broader theory of elliptic integro-differential equations developed in [33, 5, 27]. A substantial literature has recently developed on mixed local–nonlocal elliptic operators, including regularity and maximum principles [7, 19, 28, 29, 44, 45], spectral and variational problems [21, 8, 14, 35], and existence, symmetry, and critical phenomena for nonlinear elliptic equations [10, 9, 6, 34, 23, 46].

Throughout the paper, the fractional Laplacian is normalized by

$$(-\Delta)^s u(x) = C_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y = C_{n,s}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,$$

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initially for sufficiently regular functions, where

$$C_{n,s} = \frac{2^{2s}\Gamma(s+n/2)}{\pi^{n/2}|\Gamma(-s)|}.$$

We refer to [42, 18, 27] for standard background on the fractional Laplacian and elliptic integro-differential operators.

A fundamental feature of $\mathscr{L}_s$ is its lack of homogeneity under dilations. Its Fourier symbol is $|\xi|^2 + |\xi|^{2s}$. For large frequencies, the second-order term is dominant, while for small frequencies the fractional term is dominant. More precisely,

$$|\xi|^2 + |\xi|^{2s} \sim |\xi|^2 \qquad \text{as } |\xi| \to \infty,$$

whereas

$$|\xi|^2 + |\xi|^{2s} \sim |\xi|^{2s} \qquad \text{as } |\xi| \to 0.$$

This frequency transition is reflected directly in the fundamental solution. Its singularity near the pole is governed by leading order by the classical Laplacian, while its behavior at infinity is governed by leading order by the fractional Laplacian. The interaction between the two components first appears in the next term of the corresponding asymptotic expansions.

Fundamental solutions and Green kernels play a central role in potential estimates, isolated-singularity problems, and Liouville-type questions. For homogeneous local and fractional operators, the leading kernels are explicit. Green functions and fundamental solutions for various nonlocal and nonlinear nonlocal operators have been studied in [2, 26, 20]. For sums of Brownian and stable generators, sharp Green function and heat kernel estimates are available in [16, 15]. For $n \geq 3$, the exact leading-order asymptotics of the fundamental solution follow from the general results of Rao, Song, and Vondraček [41] for subordinate Brownian motions. Indeed, applying their Theorems to the Laplace exponent $\phi(\lambda) = \lambda + \lambda^s$ yields the Newtonian leading term near the origin and the fractional Riesz leading term at infinity. More recently, sharp two-sided estimates for the fundamental solution associated with the mixed operator were also established and used in the analysis of mixed Lane–Emden equations in [34].

The first purpose of this paper is to obtain information beyond such leading-order estimates. We determine the first nontrivial correction to the fundamental solution of $\mathscr{L}_s$ both near the origin and at infinity. We obtain the corresponding expansion for its gradient at the same level of precision. The structure of the correction depends on both the dimension and the fractional exponent. In particular, critical homogeneous degrees lead to logarithmic terms. Near the origin, this occurs when $n = 3$ and $s = \frac{1}{2}$. At infinity, $s = \frac{1}{2}$ is again exceptional. The second term in the low-frequency expansion is the constant multiplier -1. After multiplication by the cutoff, its inverse Fourier transform is a Schwartz function, so the following $|\xi|$-term gives the first nonzero correction in the far field.

We work mainly with distributional solutions. Set

$$\mathcal{L}^{2s}(\mathbb{R}^n) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,\mathrm{d}x < \infty \right\}.$$

If $\Omega \subset \mathbb{R}^n$ is open and $u \in \mathcal{L}^{2s}(\mathbb{R}^n)$, we define $\mathscr{L}_s u$ in Ω by

$$\langle \mathscr{L}_s u, \phi \rangle = \int_{\mathbb{R}^n} u(x) \mathscr{L}_s \phi(x)\,\mathrm{d}x, \qquad \phi \in C_c^\infty(\Omega).$$

Since $|(-\Delta)^s \phi(x)| \le C_\phi (1+|x|)^{-n-2s}$, the pairing is well defined.

We now state the first main result. The Fourier transform is normalized by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x, \qquad f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\,\mathrm{d}\xi. \tag{1.1}$$

We write

$$\omega_{n-1} = |\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad \kappa_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)}.$$

For $\alpha \notin \{n, n+2, n+4, \ldots\}$, the constant $\kappa_{n,\alpha}$ is understood through the meromorphic continuation of the Riesz-kernel family described in Appendix B.

Theorem 1.1. *Let $n \ge 2$ and $0 < s < 1$. Define*

$$\Gamma_{n,s} = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 + |\xi|^{2s}} \right) \in \mathcal{S}'(\mathbb{R}^n).$$

Then

$$\mathscr{L}_s \Gamma_{n,s} = \delta_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{1.2}$$

and $\Gamma_{n,s}$ is represented by a smooth real-valued radial function in $\mathbb{R}^n \setminus \{0\}$. Moreover, the following assertions hold.

(1) ***Asymptotics at the origin.***
$n \ge 3$ and $4 - 2s < n$: *As $|x| \to 0$,*

$$\begin{aligned} \Gamma_{n,s}(x) &= \frac{|x|^{2-n}}{(n-2)\omega_{n-1}} - \kappa_{n,4-2s}|x|^{4-2s-n} + o\big(|x|^{4-2s-n}\big), \\ \nabla\Gamma_{n,s}(x) &= -\frac{x}{\omega_{n-1}|x|^n} - (4-2s-n)\kappa_{n,4-2s} x|x|^{2-2s-n} + o\big(|x|^{3-2s-n}\big). \end{aligned} \tag{1.3}$$

$n = 3$ and $s = \frac{1}{2}$: *As $|x| \to 0$,*

$$\begin{aligned} \Gamma_{3,1/2}(x) &= \frac{1}{4\pi|x|} - \frac{1}{2\pi^2} \log\frac{1}{|x|} + O(1), \\ \nabla\Gamma_{3,1/2}(x) &= -\frac{x}{4\pi|x|^3} + \frac{1}{2\pi^2}\frac{x}{|x|^2} + O(1). \end{aligned} \tag{1.4}$$

$n = 3$ and $0 < s < \frac{1}{2}$: *There exists $C_{3,s} \in \mathbb{R}$ such that, as $|x| \to 0$,*

$$\begin{aligned} \Gamma_{3,s}(x) &= \frac{1}{4\pi|x|} + C_{3,s} - \kappa_{3,4-2s}|x|^{1-2s} + o\big(|x|^{1-2s}\big), \\ \nabla\Gamma_{3,s}(x) &= -\frac{x}{4\pi|x|^3} - (1-2s)\kappa_{3,4-2s} x|x|^{-1-2s} + o\big(|x|^{-2s}\big). \end{aligned} \tag{1.5}$$

$n=2$: *There exists $C_{2,s}\in\mathbb{R}$ such that, as $|x|\to 0$,*

$$\begin{aligned}\Gamma_{2,s}(x) &= \frac{1}{2\pi}\log\frac{1}{|x|} + C_{2,s} - \kappa_{2,4-2s}|x|^{2-2s} + o\big(|x|^{2-2s}\big)\,,\\ \nabla\Gamma_{2,s}(x) &= -\frac{x}{2\pi|x|^2} - (2-2s)\kappa_{2,4-2s}x|x|^{-2s} + o\big(|x|^{1-2s}\big)\,.\end{aligned}\tag{1.6}$$

(2) ***Asymptotics at infinity.***
$s\neq\frac{1}{2}$: *As $|x|\to\infty$,*

$$\begin{aligned}\Gamma_{n,s}(x) &= \kappa_{n,2s}|x|^{2s-n} - \kappa_{n,4s-2}|x|^{4s-2-n} + o\big(|x|^{4s-2-n}\big)\,,\\ \nabla\Gamma_{n,s}(x) &= -(n-2s)\kappa_{n,2s}\frac{x}{|x|^{n+2-2s}}\\ &\qquad + (n+2-4s)\kappa_{n,4s-2}\frac{x}{|x|^{n+4-4s}} + o\big(|x|^{4s-3-n}\big)\,.\end{aligned}\tag{1.7}$$

$s=\frac{1}{2}$: *As $|x|\to\infty$,*

$$\begin{aligned}\Gamma_{n,1/2}(x) &= \kappa_{n,1}|x|^{1-n} + \kappa_{n,-1}|x|^{-n-1} + o\big(|x|^{-n-1}\big)\,,\\ \nabla\Gamma_{n,1/2}(x) &= -(n-1)\kappa_{n,1}\frac{x}{|x|^{n+1}} - (n+1)\kappa_{n,-1}\frac{x}{|x|^{n+3}} + o\big(|x|^{-n-2}\big)\,.\end{aligned}\tag{1.8}$$

All vector-valued little-o terms are understood with respect to the Euclidean norm.

Theorem 1.1 makes the two-scale structure of $\mathscr{L}_s$ explicit. Near the origin, the leading singularity is given by the fundamental solution of $-\Delta$, namely the Newtonian kernel when $n\geq 3$ and the logarithmic kernel when $n=2$. At infinity, the leading term is the fractional Riesz kernel $\kappa_{n,2s}|x|^{2s-n}$. The displayed scale-dependent correction terms quantify the first interaction between the two parts of the operator.

The gradient expansions are also important for the applications below. For every $R>0$,

$$\Gamma_{n,s}\in W^{1,p}(B_R),\qquad \text{for } 1\leq p<\frac{n}{n-1},$$

while

$$\nabla\Gamma_{n,s}\notin L^{n/(n-1)}(B_R).\tag{1.9}$$

Thus the gradient has precisely the endpoint singularity of the classical Newtonian kernel. This borderline behavior will be used to eliminate derivatives of the Dirac mass in the analysis of isolated singularities.

The classical Bôcher theorem determines the singular part of a nonnegative harmonic function at an isolated point. More precisely, if u is nonnegative and harmonic in a punctured ball, then

$$u=a\Phi+h,$$

where $a\geq 0$, Φ is the fundamental solution of $-\Delta$, and h is harmonic in the full ball. The result first appeared in Bôcher's paper [3]. A later proof can be found in [1]. Extensions to more general second-order elliptic equations were obtained in several classical works [36, 30]. The distributional formulation of Brezis and Lions [4] is particularly relevant to the present work. In their approach, extending the equation across the puncture produces a defect distribution T satisfying $\operatorname{supp} T \subset \{0\}$. Related classification results for positive solutions of semilinear elliptic equations are given in [31, 32]. For the fractional Laplacian, maximum principles and Bôcher-type theorems on punctured balls were established in [40]. A distributional theory for nonnegative solutions of fractional Laplace equations with an isolated singularity was developed further in [39]. More generally, Klimsiak [38] recently proved a Bôcher-type theorem within a broad framework of elliptic equations with drift-perturbed Lévy operators, which includes the mixed operator considered here. His theorem is formulated for functions that are nonnegative on all of $\mathbb{R}^n$, whereas Theorem 1.2 below requires nonnegativity only in B_1. His argument is based on the probabilistic potential theory. In the present paper, we instead give a direct analytic proof based on the precise asymptotic expansions of $\Gamma_{n,s}$ and $\nabla\Gamma_{n,s}$. Once the point-supported defect is shown to have order at most one, it may still contain the first derivatives of δ_0. The Newtonian leading term of $\nabla\Gamma_{n,s}$ in Theorem 1.1 has exactly the endpoint non-integrability (1.9) needed to exclude these dipole terms under the nonnegativity assumption.

Our second main result is the following distributional Bôcher theorem.

Theorem 1.2 (Bôcher theorem for $-\Delta + (-\Delta)^s$)**.** *Let $B_1 \subset \mathbb{R}^n$, $n \geq 2$, and $s \in (0,1)$. Assume $u \in \mathcal{L}^{2s}(\mathbb{R}^n)$ satisfies $u \geq 0$ a.e. in B_1 and*

$$\mathscr{L}_s u + cu \geq 0 \qquad \text{in } \mathcal{D}'(B_1 \setminus \{0\}),$$

where $c \in L^\infty(B_1)$. Then there exist $a \geq 0$ and a nonnegative Radon measure μ in B_1, with $\mu(\{0\}) = 0$, such that

$$\mathscr{L}_s u + cu = \mu + a\delta_0 \qquad \text{in } \mathcal{D}'(B_1).$$

In particular, the entire point-supported singular part of a nonnegative distributional supersolution consists of a nonnegative multiple of δ_0. No derivative of the Dirac mass can occur. The theorem is formulated under the natural global tail condition $u \in \mathcal{L}^{2s}(\mathbb{R}^n)$, which is needed in order to define the nonlocal part distributionally.

Maximum principles for local, nonlocal, and mixed elliptic equations have been studied in many different settings. Representative results can be found in [37, 17, 7, 45]. The relation between maximum principles on punctured domains and Bôcher-type theorems for the fractional Laplacian is developed in [40, 39]. In the present mixed setting, Theorem 1.2 provides the mechanism that allows the puncture to be recovered distributionally. After replacing the zero-order coefficient by its positive part, the singular contribution at the puncture has the favorable sign. This reduces the problem to a quantitative minimum estimate for a global supersolution on the ball.

We obtain the following two principles.

Theorem 1.3 (Positive maximum principle)**.** *Let $n \geq 2$, $s \in (0,1)$, $B_r(x_0) \subset \mathbb{R}^n$, and $c \in L^\infty(B_r(x_0))$. Assume $u \in \mathcal{L}^{2s}(\mathbb{R}^n)$, $u \geq 0$ a.e. in $\mathbb{R}^n$, and*

$$\mathscr{L}_s u + cu \geq 0 \qquad \text{in } \mathcal{D}'(B_r(x_0) \setminus \{x_0\}).$$

If

$$u \geq m > 0 \qquad a.e.\ \ in\ B_r(x_0) \setminus B_{r/2}(x_0),$$

then

$$u \geq \alpha m \qquad a.e.\ \ in\ B_r(x_0) \setminus \{x_0\},$$

where

$$\alpha = \alpha\big(n, s, r^{2s}\|c^+\|_{L^\infty(B_r(x_0))}\big) > 0, \qquad c^+ = \max\{c, 0\}.$$

In particular, if $0 < r \leq 1$, then α may be chosen to depend only on n, s, and $\|c^+\|_{L^\infty(B_r(x_0))}$. If $c \leq 0$, it may be chosen independently of both r and c.

Theorem 1.4 (Antisymmetric maximum principle)**.** *Let $n \geq 2$, $s \in (0,1)$,*

$$H = \{x_1 > 0\}, \qquad \widetilde{x} = (-x_1, x'),$$

and let $B_r(x_0) \subset H$ and $c \in L^\infty(B_r(x_0))$. Assume $u \in \mathcal{L}^{2s}(\mathbb{R}^n)$ is antisymmetric with respect to ∂H, namely

$$u(\widetilde{x}) = -u(x) \qquad for\ a.e.\ \ x \in H,$$

and $u \geq 0$ a.e. in H. Suppose

$$\mathscr{L}_s u + cu \geq 0 \qquad in\ \mathcal{D}'(B_r(x_0) \setminus \{x_0\}),$$

and

$$u \geq m > 0 \qquad a.e.\ \ in\ B_r(x_0) \setminus B_{r/2}(x_0).$$

Then

$$u \geq \alpha m \qquad a.e.\ \ in\ B_r(x_0) \setminus \{x_0\},$$

where

$$\alpha = \alpha\big(n, s, r^{2s}\|c^+\|_{L^\infty(B_r(x_0))}\big) > 0.$$

As above, for $0 < r \leq 1$ the constant may be chosen uniformly in r, while for $c \leq 0$ it depends only on n and s.

The quantity $r^{2s}\|c^+\|_{L^\infty(B_r(x_0))}$ appearing in the two estimates is natural. At an interior minimum the local part has the favorable sign. The contribution of the fractional operator from an annulus whose scale is r is of order r^{-2s}. The competition with the zero-order term therefore occurs through the dimensionless quantity $r^{2s}\|c^+\|_\infty$. The same scaling enters the antisymmetric argument after the fractional kernel is paired with its reflection across the boundary hyperplane.

We briefly explain the main ideas of the proofs. For Theorem 1.1, we start from the multiplier

$$m(\xi) = \frac{1}{|\xi|^2 + |\xi|^{2s}}$$

and separate low and high frequencies. At high frequencies,

$$m(\xi) = |\xi|^{-2} - |\xi|^{-(4-2s)} + O\Big(|\xi|^{-(6-4s)}\Big),$$

while at low frequencies,

$$m(\xi) = |\xi|^{-2s} - |\xi|^{2-4s} + O\big(|\xi|^{4-6s}\big).$$

The homogeneous terms are inverted using the Fourier transform formulas for Riesz kernels and their distributional continuation. When a homogeneous exponent reaches a critical degree, the corresponding distributional continuation produces the logarithmic contribution. Cutoff multipliers and dyadic kernel estimates control the remainder terms together with their derivatives.

The proof of Theorem 1.2 follows the distributional strategy of [40, Theorems 3 and 4], adapted to the mixed operator. A radial barrier first shows that the positive distribution on the punctured ball has finite mass near the origin. After this measure is extended across the puncture, the remaining defect is a distribution supported at the origin. A cutoff argument shows that its order is at most one, and hence it has the form $a\delta_0+d\cdot\nabla\delta_0$. A localized potential representation, together with the critical endpoint non-integrability of the dipole term furnished by the gradient expansion in Theorem 1.1, excludes $d\neq 0$. An averaged estimate near the origin then implies $a\geq 0$.

The two maximum principles are obtained by applying Theorem 1.2 to restore the supersolution inequality across the puncture. After mollification, the resulting inequality can be evaluated at an interior minimum. In the antisymmetric case, the argument also uses the strict positivity of the reflected kernel

$$\frac{1}{|x-y|^{n+2s}}-\frac{1}{|x-\widetilde{y}|^{n+2s}}>0, \qquad x,y\in H.$$

The paper is organized as follows. Section 2 is devoted to the construction of the fundamental solution and the proof of its asymptotic expansions in Theorem 1.1. Section 3 establishes the Bôcher-type theorem. The positive and antisymmetric maximum principles are proved in Section 4. The appendices contain the auxiliary results used in these arguments, including distributional properties of the mixed operator, Fourier transform formulas for homogeneous distributions, and the cutoff and dyadic kernel estimates required for the asymptotic analysis.

2. Proof of Theorem 1.1

Proof. **Step 1. Construction and frequency decomposition.** Let $m(\xi)=(|\xi|^2+|\xi|^{2s})^{-1}$. Since $m(\xi)\asymp|\xi|^{-2s}$ as $|\xi|\to 0$, $m(\xi)\asymp|\xi|^{-2}$ as $|\xi|\to\infty$, and $n>2s$, the function m belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and defines a tempered distribution. For every $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\langle\Gamma_{n,s},\varphi\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}m(\xi)\widehat{\varphi}(-\xi)\,\mathrm{d}\xi,$$

and the integral is absolutely convergent. Moreover,

$$|\xi|^2m(\xi),\ |\xi|^{2s}m(\xi)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

with at most polynomial growth, and

$$\big(|\xi|^2+|\xi|^{2s}\big)m(\xi)=1$$

almost everywhere. Hence (1.2) follows in $\mathcal{S}'(\mathbb{R}^n)$.

Choose a radial cutoff $\chi \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\chi = 1 \quad \text{in } \{|\xi| \le 1\}, \qquad \chi = 0 \quad \text{in } \{|\xi| \ge 2\},$$

and set $\eta = 1 - \chi$. Then

$$\Gamma_{n,s} = \mathcal{F}^{-1}(\chi m) + \mathcal{F}^{-1}(\eta m). \tag{2.1}$$

For every multi-index β, $\xi^\beta \chi(\xi) m(\xi) \in L^1$. Hence $\mathcal{F}^{-1}(\chi m) \in C^\infty(\mathbb{R}^n)$. For every multi-index γ,

$$|\partial_\xi^\gamma (\eta m)(\xi)| \le C_\gamma |\xi|^{-2-|\gamma|}, \quad \text{for any } |\xi| \ge 1.$$

Lemma C.1 therefore shows that $\mathcal{F}^{-1}(\eta m)$ is smooth away from the origin and rapidly decreasing at infinity. Since m, χ, and η are real-valued and radial, both terms in (2.1) are real-valued and radial.

Step 2. The high-frequency expansion. The exact identity

$$m(\xi) = |\xi|^{-2} - |\xi|^{-(4-2s)} + R_\infty(\xi), \qquad R_\infty(\xi) = \frac{|\xi|^{-(6-4s)}}{1 + |\xi|^{2s-2}}, \tag{2.2}$$

follows from $(1+z)^{-1} = 1 - z + z^2(1+z)^{-1}$, with $z = |\xi|^{2s-2}$. For every multi-index ρ,

$$|\partial_\xi^\rho R_\infty(\xi)| \le C_\rho |\xi|^{-(6-4s)-|\rho|} \qquad \text{for any } |\xi| \ge 1. \tag{2.3}$$

Indeed, $|\xi|^{2s-2} \le 1$ in this region, and differentiation of radial powers gives $|\partial_\xi^\rho |\xi|^a| \le C_{a,\rho} |\xi|^{a-|\rho|}$. The Leibniz and chain rules then give (2.3).

Step 3. The case $n \ge 3$ and $4 - 2s < n$. Corollary C.2 yields

$$\mathcal{F}^{-1}(\eta |\xi|^{-2})(x) = \frac{|x|^{2-n}}{(n-2)\omega_{n-1}} + H_0(x), \tag{2.4}$$

$$\mathcal{F}^{-1}(\eta |\xi|^{-(4-2s)})(x) = \kappa_{n,4-2s} |x|^{4-2s-n} + H_1(x), \tag{2.5}$$

where H_0, H_1 are smooth and radial near the origin. The remaining smooth contribution $\mathcal{F}^{-1}(\chi m)$ has the same property. Thus these regular terms are bounded, and their gradients are bounded.

Set $\mu = 6 - 4s$. By (2.3) and Lemma C.1,

$$\mathcal{F}^{-1}(\eta R_\infty)(x) = \begin{cases} O(|x|^{\mu-n}), & \mu < n, \\ O(1 + |\log|x||), & \mu = n, \\ O(1), & \mu > n, \end{cases} \qquad |x| \to 0.$$

If $\mu < n$, then

$$\frac{|x|^{\mu-n}}{|x|^{4-2s-n}} = |x|^{2-2s} \to 0.$$

If $\mu \ge n$, then by $4 - 2s - n < 0$, so logarithmic and bounded terms are also $o(|x|^{4-2s-n})$. Hence

$$\mathcal{F}^{-1}(\eta R_\infty)(x) = o\big(|x|^{4-2s-n}\big). \tag{2.6}$$

Applying the differentiated estimate in Lemma C.1 gives

$$\nabla \mathcal{F}^{-1}(\eta R_\infty)(x) = \begin{cases} O(|x|^{\mu-n-1}), & \mu < n+1, \\ O(1+|\log|x||), & \mu = n+1, \\ O(1), & \mu > n+1. \end{cases}$$

In the first case the quotient by $|x|^{3-2s-n}$ is again $|x|^{2-2s}$, in the remaining cases we use $3-2s-n<0$. Therefore

$$\nabla \mathcal{F}^{-1}(\eta R_\infty)(x) = o\big(|x|^{3-2s-n}\big)\,. \tag{2.7}$$

Combining (2.1), (2.2), (2.4), (2.5), (2.6), and (2.7) proves (1.3). The derivative of the second homogeneous term is

$$\nabla\big(-\kappa_{n,4-2s}|x|^{4-2s-n}\big) = -(4-2s-n)\kappa_{n,4-2s}x|x|^{2-2s-n}.$$

Step 4. The case $n=3$ and $s=\frac{1}{2}$. Retaining one additional term gives the exact decomposition

$$m(\xi) = |\xi|^{-2} - |\xi|^{-3} + |\xi|^{-4} - \frac{|\xi|^{-5}}{1+|\xi|^{-1}}. \tag{2.8}$$

The first term contributes $(4\pi|x|)^{-1}$, modulo a smooth radial function. Since η vanishes near the origin, $\eta|\xi|^{-3} = \eta\,\mathrm{Fp}\,|\xi|^{-3}$, and (B.2) gives

$$\mathcal{F}^{-1}(\eta|\xi|^{-3})(x) = \frac{1}{2\pi^2}\log\frac{1}{|x|} + H_2(x),$$

where H_2 is smooth and radial near the origin. Moreover,

$$\mathcal{F}^{-1}(\eta|\xi|^{-4})(x) = \kappa_{3,4}|x| + H_3(x), \qquad \kappa_{3,4} = -\frac{1}{8\pi},$$

so this term and its gradient are bounded. The last multiplier in (2.8) is a symbol of order -5, since $5>3+1$, its inverse Fourier transform and its gradient are bounded near the origin. The signs in (2.8) prove (1.4).

Step 5. The case $n=3$ and $0<s<\frac{1}{2}$. Here $4-2s\in(3,4)$, so this exponent is nonexceptional. Hence

$$\mathcal{F}^{-1}(\eta|\xi|^{-(4-2s)})(x) = \kappa_{3,4-2s}|x|^{1-2s} + H_4(x), \tag{2.9}$$

where H_4 is smooth and radial. The remainder ηR_∞ is radial, has order $-(6-4s)$, and

$$0 < 1-2s < \min\{(6-4s)-3, 2\}.$$

The radial regularity statement in Lemma C.1, with $\beta = 1-2s$, gives

$$\mathcal{F}^{-1}(\eta R_\infty)(x) = \mathcal{F}^{-1}(\eta R_\infty)(0) + o(|x|^{1-2s}), \qquad \nabla\mathcal{F}^{-1}(\eta R_\infty)(x) = o(|x|^{-2s}).$$

Every smooth radial function H satisfies

$$H(x) = H(0) + O(|x|^2), \qquad \nabla H(x) = O(|x|).$$

Collecting all constant terms into $C_{3,s}$ and using (2.2) and (2.9) proves (1.5).

Step 6. The case $n = 2$. The leading term is critical. By (C.8), we have

$$\mathcal{F}^{-1}(\eta|\xi|^{-2})(x) = \frac{1}{2\pi}\log\frac{1}{|x|} + H_5(x),$$

with H_5 smooth and radial. Since $4 - 2s \in (2, 4)$ is nonexceptional,

$$\mathcal{F}^{-1}(\eta|\xi|^{-(4-2s)})(x) = \kappa_{2,4-2s}|x|^{2-2s} + H_6(x).$$

Furthermore,

$$0 < 2 - 2s < \min\{(6 - 4s) - 2, 2\}.$$

Applying the radial regularity statement in Lemma C.1 with $\beta = 2 - 2s$, and using $H(x) = H(0) + O(|x|^2)$, $\nabla H(x) = O(|x|)$ for smooth radial functions, gives (1.6) after the constant terms are collected into $C_{2,s}$.

Step 7. The case $|x| \to \infty$ and $s \neq \frac{1}{2}$. The exact low-frequency expansion is

$$m(\xi) = |\xi|^{-2s} - |\xi|^{2-4s} + R_0(\xi), \qquad R_0(\xi) = \frac{|\xi|^{4-6s}}{1 + |\xi|^{2-2s}}. \tag{2.10}$$

For every multi-index ρ,

$$|\partial_\xi^\rho R_0(\xi)| \le C_\rho|\xi|^{4-6s-|\rho|} \qquad \text{for any } 0 < |\xi| \le 2.$$

Since $4s - 2 \in (-2, 2) \setminus \{0\}$, Corollary C.2 yields, for every $N > 0$,

$$\mathcal{F}^{-1}(\chi|\xi|^{-2s})(x) = \kappa_{n,2s}|x|^{2s-n} + O(|x|^{-N}),$$
$$\mathcal{F}^{-1}(\chi|\xi|^{2-4s})(x) = \kappa_{n,4s-2}|x|^{4s-2-n} + O(|x|^{-N}).$$

Because $4 - 6s > -n$, the low-frequency estimate in Lemma C.1 gives

$$\mathcal{F}^{-1}(\chi R_0)(x) = O(|x|^{6s-4-n}), \qquad \nabla\mathcal{F}^{-1}(\chi R_0)(x) = O(|x|^{6s-5-n}).$$

The ratios to the respective second terms are

$$\frac{|x|^{6s-4-n}}{|x|^{4s-2-n}} = \frac{|x|^{6s-5-n}}{|x|^{4s-3-n}} = |x|^{2s-2} \to 0.$$

The high-frequency contribution $\mathcal{F}^{-1}(\eta m)$ is rapidly decreasing. Combining these estimates with (2.10) proves (1.7).

Step 8. The case $|x| \to \infty$ and $s = \frac{1}{2}$. The finite identity

$$m(\xi) = |\xi|^{-1} - 1 + |\xi| - \frac{|\xi|^2}{1 + |\xi|} \tag{2.11}$$

replaces (2.10). For every $N > 0$,

$$\mathcal{F}^{-1}(\chi|\xi|^{-1})(x) = \kappa_{n,1}|x|^{1-n} + O(|x|^{-N}),$$

and

$$\mathcal{F}^{-1}(\chi|\xi|)(x) = \kappa_{n,-1}|x|^{-n-1} + O(|x|^{-N}).$$

The inverse Fourier transform of χ is a Schwartz function. The last multiplier in (2.11) satisfies the low-frequency symbol estimates with exponent 2. Hence

$$\mathcal{F}^{-1}\left(\chi\frac{|\xi|^2}{1+|\xi|}\right)(x) = O(|x|^{-n-2}), \qquad \nabla\mathcal{F}^{-1}\left(\chi\frac{|\xi|^2}{1+|\xi|}\right)(x) = O(|x|^{-n-3}).$$

Together with the rapid decay of $\mathcal{F}^{-1}(\eta m)$, this proves (1.8). Notice that every gradient remainder used above comes directly from the differentiated symbol estimates in Lemma C.1. In particular, no undifferentiated little-o term is differentiated. $\square$

3. Proof of Theorem 1.2

Proof. Put $\mathscr{L}_s = -\Delta + (-\Delta)^s$ and $T = \mathscr{L}_s u + cu \in \mathcal{D}'(B_1 \setminus \{0\})$. Since $T \geq 0$, it is a nonnegative Radon measure on $B_1 \setminus \{0\}$, which we denote by μ. We extend μ to B_1 by $\mu(\{0\}) = 0$. The argument below follows the distributional scheme of [40, Theorems 3 and 4], with the mixed local–nonlocal estimates made explicit.

Step 1. μ has finite mass near 0. Set $M = \|c\|_{L^\infty(B_1)}$. Choose a radial $\eta \in C_c^\infty(B_2)$ with $\eta = 1$ in B_1, and fix $0 < \beta < 2s$ and $0 < \alpha < 1$. For $0 < \varepsilon < e^{-1}$ define

$$\phi_\varepsilon(r) = \begin{cases} 1 + \eta(2r)\left[r^\beta - \left(\dfrac{\log r}{\log\varepsilon}\right)^\alpha\right], & n = 2, \\ 1 + \eta(r) r^\beta - \left(\dfrac{\varepsilon}{r}\right)^\alpha, & n \geq 3. \end{cases} \tag{3.1}$$

For $r < 1/4$ the cutoffs in (3.1) are equal to one. If $n \geq 3$, then

$$\begin{aligned} -\Delta\phi_\varepsilon(r) &= -\beta(\beta+n-2)r^{\beta-2} - \alpha(n-2-\alpha)\varepsilon^\alpha r^{-\alpha-2}, \\ |(-\Delta)^s\phi_\varepsilon(r)| &\leq Cr^{\beta-2s} + C\varepsilon^\alpha r^{-\alpha-2s}. \end{aligned} \tag{3.2}$$

Indeed, since $0 < \alpha < 1 < n - 2s$, Proposition B.1 gives

$$(-\Delta)^s(|x|^{-\alpha}) = C_{n,s,\alpha}|x|^{-\alpha-2s} \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

It remains to estimate $F(x) = \eta(|x|)|x|^\beta$. For $r = |x| < 1/4$, the symmetric integral formula gives

$$(-\Delta)^s F(x) = \frac{C_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2F(x) - F(x+z) - F(x-z)}{|z|^{n+2s}}\,\mathrm{d}z.$$

Since F is smooth in $B_{r/2}(x)$,

$$|2F(x) - F(x+z) - F(x-z)| \leq \begin{cases} Cr^{\beta-2}|z|^2, & |z| < r/2, \\ C|z|^\beta, & r/2 \leq |z| \leq 3, \\ C, & |z| > 3. \end{cases}$$

Because $0 < \beta < 2s < 2$, integration in the three regions yields $|(-\Delta)^s F(x)| \leq Cr^{\beta-2s}$. Together with the estimate for $|x|^{-\alpha}$, this proves (3.2), with C independent of ε.

If $n=2$, let $L=\log(1/r)$ and $E=\log(1/\varepsilon)$. Then

$$-\Delta\phi_\varepsilon(r) = -\beta^2 r^{\beta-2} - \alpha(1-\alpha)E^{-\alpha}L^{\alpha-2}r^{-2},$$
$$|(-\Delta)^s\phi_\varepsilon(r)| \le Cr^{\beta-2s} + CE^{-\alpha}r^{-2s}(1+L^\alpha).$$

Indeed, for $F(x)=\eta(2|x|)(\log(1/|x|))^\alpha$, splitting the defining integral into $B_{r/2}(x)$ and its complement gives

$$|(-\Delta)^s F(x)| \le Cr^{-2s}(1+L^\alpha), \qquad r=|x|<\frac{1}{4}.$$

Since

$$r^{2-2s}\to 0, \qquad r^{2-2s}(L^{2-\alpha}+L^2)\to 0, \quad \text{as } r\downarrow 0,$$

the preceding estimates imply that there exists $r_0\in(0,1/8)$, independent of ε, such that

$$(\mathscr{L}_s+M)\phi_\varepsilon \le 0 \quad \text{in } \mathcal{D}'(B_{2r_0}).$$

Set $\psi_\varepsilon=(\phi_\varepsilon)^+$. Proposition A.1 gives

$$(\mathscr{L}_s+M)\psi_\varepsilon \le 0 \quad \text{in } \mathcal{D}'(B_{2r_0}).$$

Setting $\psi_\varepsilon^\delta=\psi_\varepsilon * j_\delta$, Lemma A.2 gives, for all sufficiently small $\delta>0$,

$$(\mathscr{L}_s+M)\psi_\varepsilon^\delta \le 0 \quad \text{in } B_{r_0}.$$

Choose $h\in C_c^\infty(B_{3r_0/4})$ with $0\le h\le 1$ and $h=1$ in $B_{r_0/2}$. Taking δ smaller if necessary, $h\psi_\varepsilon^\delta\in C_c^\infty(B_1\setminus\{0\})$. Put $\psi=\psi_\varepsilon^\delta$. By the product identity

$$\begin{aligned}\mathscr{L}_s(h\psi)(x) &= h(x)\mathscr{L}_s\psi(x)+\psi(x)\mathscr{L}_s h(x) - 2\nabla h(x)\cdot\nabla\psi(x)\\ &\quad - C_{n,s}\int_{\mathbb{R}^n}\frac{(h(x)-h(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y,\end{aligned}$$

we have

$$\begin{aligned}0 &\le \int_{B_1} h\psi\,\mathrm{d}\mu = T(h\psi)\\ &= \int_{B_1} u\,\mathscr{L}_s(h\psi)\,\mathrm{d}x + \int_{B_1} cuh\psi\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} u\,\mathscr{L}_s(h\psi)\,\mathrm{d}x\\ &\le \int_{B_1} u\psi\,\mathscr{L}_s h\,\mathrm{d}x - 2\int_{B_1} u\,\nabla h\cdot\nabla\psi\,\mathrm{d}x\\ &\quad - C_{n,s}\int_{B_1} u(x)\int_{\mathbb{R}^n}\frac{(h(x)-h(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} u\,\mathscr{L}_s(h\psi)\,\mathrm{d}x.\end{aligned}$$

where the last inequality uses the fact that

$$\int_{B_1} uh\big[(\mathscr{L}_s+M)\psi\big]\,\mathrm{d}x + \int_{B_1}(c-M)uh\psi\,\mathrm{d}x \le 0.$$

Uniformly in ε and δ,

$$
\begin{aligned}
&0 \le \psi \le C, \qquad |\nabla \psi| \le C \quad \text{on } \operatorname{supp} \nabla h, \\
&\sup_{x \in B_1} \left| \int_{\mathbb{R}^n} \frac{(h(x)-h(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}} \, \mathrm{d}y \right| \le C, \\
&|\mathscr{L}_s(h\psi)(x)| \le \frac{C}{1+|x|^{n+2s}}, \qquad x \in \mathbb{R}^n \setminus B_1.
\end{aligned}
\tag{3.3}
$$

Indeed, in (3.3) either h is constant near x, or both differences are $O(|x-y|)$. Hence, the local singularity is $|x-y|^{-n-2s+2} \in L^1_{\mathrm{loc}}$ because $s<1$. To make the bound uniform, let $K = \operatorname{supp} \nabla h$. Since $K \Subset B_{r_0} \setminus \{0\}$, choose $\rho > 0$ so that the 3ρ-neighborhood of K is contained in $B_{r_0} \setminus \{0\}$. The explicit barriers and mollification give

$$
\|\nabla \psi_\varepsilon^\delta\|_{L^\infty(\{\operatorname{dist}(x,K)<3\rho\})} \le C.
$$

If $\operatorname{dist}(x,K) \ge 2\rho$, then h is constant on $B_\rho(x)$. Otherwise, when $|x-y|<\rho$, both factors in the numerator are bounded by $C|x-y|$. Hence, the local part is bounded by $C\int_{|z|<\rho} |z|^{-n-2s+2} \, \mathrm{d}z$. In $|x-y| \ge \rho$, the uniform bounds of h and ψ_ε^δ give the bound $C\int_{|z|\ge\rho} |z|^{-n-2s} \, \mathrm{d}z$. This proves (3.3) with a constant independent of ε and δ. Consequently,

$$
0 \le \int_{B_{r_0/2}} \psi_\varepsilon^\delta \, \mathrm{d}\mu \le C \left(\int_{B_1} u \, \mathrm{d}x + \int_{\mathbb{R}^n \setminus B_1} \frac{|u(x)|}{1+|x|^{n+2s}} \, \mathrm{d}x \right) \le C.
\tag{3.4}
$$

For fixed ε, $\psi_\varepsilon^\delta \to \psi_\varepsilon$ locally uniformly as $\delta \downarrow 0$, so Fatou's lemma and (3.4) give $\int_{B_{r_0/2}} \psi_\varepsilon \, \mathrm{d}\mu \le C$. On $0<|x|<r_0/2$ we have $\psi_\varepsilon(x) \to 1+|x|^\beta$ as $\varepsilon \downarrow 0$; a second application of Fatou's lemma yields $\mu(B_{r_0/2}) < \infty$. Hence μ is a Radon measure on B_1.

Step 2. The singular distribution has order at most one. Set $S = \mathscr{L}_s u + cu - \mu \in \mathcal{D}'(B_1)$, so $\operatorname{supp} S \subset \{0\}$. Let $H \in C_c^\infty(B_1)$ satisfy $H(0)=0$ and $\nabla H(0) = 0$. Choose $\rho \in C_c^\infty(B_2)$ with $\rho = 1$ in B_1 and put $H_\varepsilon(x) = \rho(x/\varepsilon) H(x)$. Since $S(H) = S(H_\varepsilon)$,

$$
|H_\varepsilon| \le C\varepsilon^2, \qquad \operatorname{supp} H_\varepsilon \subset B_{2\varepsilon}, \qquad \|D^2 H_\varepsilon\|_{L^\infty} \le C.
$$

Writing $H_\varepsilon(x) = \varepsilon^2 G_\varepsilon(x/\varepsilon)$, with $\|G_\varepsilon\|_{C^2} \le C$, gives

$$
|(-\Delta)^s H_\varepsilon(x)| \le \begin{cases} C\varepsilon^{2-2s}, & |x| < 4\varepsilon, \\ C\varepsilon^{n+2}|x|^{-n-2s}, & |x| \ge 4\varepsilon. \end{cases}
$$

Therefore

$$
\begin{aligned}
|S(H)| = |S(H_\varepsilon)| \le C \int_{B_{2\varepsilon}} |u| \, \mathrm{d}x + C\varepsilon^{2-2s} \int_{B_{4\varepsilon}} |u| \, \mathrm{d}x \\
+ C\varepsilon^{n+2} \int_{\mathbb{R}^n \setminus B_{4\varepsilon}} \frac{|u(x)|}{|x|^{n+2s}} \, \mathrm{d}x + C\varepsilon^2 \int_{B_{2\varepsilon}} |cu| \, \mathrm{d}x + C\varepsilon^2 \mu(B_{2\varepsilon}) \to 0.
\end{aligned}
$$

Here, for the third term, we split $4\varepsilon < |x| < 1$ and $|x| \ge 1$ and use

$$
\varepsilon^{n+2} \int_{4\varepsilon < |x| < 1} \frac{|u(x)|}{|x|^{n+2s}} \, \mathrm{d}x \le C\varepsilon^{2-2s} \int_{B_1} |u| \, \mathrm{d}x.
$$

Hence
$$S(H) = 0 \quad \text{whenever} \quad H(0) = |\nabla H(0)| = 0.$$
Let $\zeta \in C_c^\infty(B_1)$ satisfy $\zeta = 1$ near 0, and define
$$a = S(\zeta) = \int_{\mathbb{R}^n} u\,\mathscr{L}_s\zeta\,\mathrm{d}x + \int_{B_1} cu\zeta\,\mathrm{d}x - \int_{B_1}\zeta\,\mathrm{d}\mu,$$
$$d_j = -S(x_j\zeta) = -\int_{\mathbb{R}^n} u\,\mathscr{L}_s(x_j\zeta)\,\mathrm{d}x - \int_{B_1} cu\,x_j\zeta\,\mathrm{d}x + \int_{B_1} x_j\zeta\,\mathrm{d}\mu.$$
For any $\varphi \in C_c^\infty(B_1)$, set $H_\varphi = \varphi - (\varphi(0) + \nabla\varphi(0)\cdot x)\zeta$. It satisfies $H_\varphi(0) = |\nabla H_\varphi(0)| = 0$. Thus
$$S(\varphi) = a\varphi(0) - d\cdot\nabla\varphi(0),$$
and therefore
$$\mathscr{L}_s u + cu = \mu + a\delta_0 + d\cdot\nabla\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{3.5}$$

Step 3. We show that $d = 0$. Let $\Gamma = \Gamma_{n,s}$ be the fundamental solution of $\mathscr{L}_s$. Set $w = u - a\Gamma - d\cdot\nabla\Gamma$ and write $\mathrm{d}\nu = \mathrm{d}\mu - cu\,\mathrm{d}x$. Then
$$\mathscr{L}_s w = \nu \quad \text{in } \mathcal{D}'(B_1).$$
Choose $\xi \in C_c^\infty(B_{7/8})$, $\xi = 1$ in $B_{3/4}$, and define
$$v(x) = \int_{B_{7/8}} \Gamma(x-y)\xi(y)\,\mathrm{d}\nu(y). \tag{3.6}$$
By Theorem 1.1, $\Gamma, \nabla\Gamma \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $1 < p < n/(n-1)$. Moreover
$$\int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}\,\mathrm{d}x \le |\nu|(B_{7/8}) \sup_{|y|<7/8} \int_{\mathbb{R}^n} \frac{|\Gamma(x-y)|}{1+|x|^{n+2s}}\,\mathrm{d}x < \infty.$$
The asymptotics in Theorem 1.1 also give $\Gamma, \nabla\Gamma \in \mathcal{L}^{2s}(\mathbb{R}^n)$; hence $w - v \in \mathcal{L}^{2s}(\mathbb{R}^n)$ and satisfies
$$\mathscr{L}_s(w - v) = 0 \quad \text{in } \mathcal{D}'(B_{3/4}).$$
Theorem A.3 gives $w - v \in C^\infty(B_{3/4})$ and hence $w \in W^{1,p}(B_{1/2})$ for every $1 < p < \frac{n}{n-1}$. Choosing any such $p > 1$, its Sobolev conjugate satisfies $p^* = np/(n-p) > n/(n-1)$. Thus, Sobolev embedding, together with $\Gamma \in L^{n/(n-1)}(B_{1/2})$, gives $w + a\Gamma \in L^{n/(n-1)}(B_{1/2})$. If $d \neq 0$, Theorem 1.1 gives, as $|x| \to 0$,
$$d\cdot\nabla\Gamma(x) = -\frac{d\cdot x}{\omega_{n-1}|x|^n} + o(|x|^{1-n}).$$
Set
$$\mathcal{C}_d = \left\{x \neq 0 : d\cdot x \ge \frac{1}{2}|d|\,|x|\right\},$$
then there exists $r_d > 0$ such that
$$(d\cdot\nabla\Gamma)^-(x) \ge \frac{|d|}{4\omega_{n-1}}|x|^{1-n}, \qquad x \in \mathcal{C}_d, \quad 0 < |x| < r_d.$$

and hence

$$(d \cdot \nabla\Gamma)^- \notin L^{n/(n-1)}(B_{1/2}).$$

On the other hand,

$$d \cdot \nabla\Gamma = u - (w + a\Gamma), \qquad (d \cdot \nabla\Gamma)^- \leq u^- + |w + a\Gamma| = |w + a\Gamma| \in L^{n/(n-1)}(B_{1/2}),$$

which yields a contradiction. Thus $d = 0$.

Step 4. Finally, we prove that $a \geq 0$. Now

$$\mathscr{L}_s u = \nu + a\delta_0, \qquad \nu = \mu - cu\,\mathrm{d}x, \qquad |\nu|(\{0\}) = 0.$$

With v as in (3.6),

$$u = v + a\Gamma + H \quad \text{in } B_{1/2}, \qquad H \in C^\infty(B_{1/2}). \tag{3.7}$$

Assume first $n \geq 3$. By Theorem 1.1, we have $|\Gamma(z)| \leq C|z|^{2-n}$ in B_1. Hence, for $0 < \delta < \varepsilon/4 < 1/8$, (3.6) gives

$$\begin{aligned}\frac{1}{\delta^2}\int_{B_\delta} |v(x)|\,\mathrm{d}x &\leq \frac{C}{\delta^2}\int_{B_{2\varepsilon}}\int_{B_\delta} |x-y|^{2-n}\,\mathrm{d}x\,\mathrm{d}|\nu|(y) \\ &\quad + \frac{C}{\delta^2}\int_{B_{7/8}\setminus B_{2\varepsilon}}\int_{B_\delta} |x-y|^{2-n}\,\mathrm{d}x\,\mathrm{d}|\nu|(y) \\ &\leq C|\nu|(B_{2\varepsilon}) + C\left(\frac{\delta}{\varepsilon}\right)^{n-2}|\nu|(B_{7/8}),\end{aligned}$$

where we use

$$\sup_{y\in\mathbb{R}^n}\int_{B_\delta} |x-y|^{2-n}\,\mathrm{d}x \leq C\delta^2.$$

Fixing ε and letting $\delta \downarrow 0$ in the preceding bound gives $\limsup_{\delta\downarrow 0} \delta^{-2}\int_{B_\delta}|v|\,\mathrm{d}x \leq C|\nu|(B_{2\varepsilon})$. Since $|\nu|(\{0\}) = 0$, letting $\varepsilon \downarrow 0$ yields

$$\frac{1}{\delta^2}\int_{B_\delta}|v|\,\mathrm{d}x \to 0 \qquad \text{as } \delta \downarrow 0.$$

Theorem 1.1 and $H \in C^\infty(B_{1/2})$ also give

$$\frac{1}{\delta^2}\int_{B_\delta}\Gamma(x)\,\mathrm{d}x \to \frac{1}{2(n-2)}, \qquad \frac{1}{\delta^2}\int_{B_\delta}|H|\,\mathrm{d}x \to 0, \quad \text{as } \delta \to 0. \tag{3.8}$$

If $a < 0$, (3.7)–(3.8) imply

$$\limsup_{\delta\downarrow 0}\frac{1}{\delta^2}\int_{B_\delta} u\,\mathrm{d}x \leq \frac{a}{2(n-2)} < 0,$$

contrary to $u \geq 0$.

For $n = 2$, by Theorem 1.1, we have $|\Gamma(z)| \leq C(\log\frac{1}{|z|} + 1)$ in B_1. Hence, for $0 < \delta < \varepsilon/4$,

$$\frac{1}{\delta^2\log(1/\delta)}\int_{B_\delta}|v(x)|\,\mathrm{d}x \leq C|\nu|(B_{2\varepsilon}) + C\frac{1+\log(1/\varepsilon)}{\log(1/\delta)}|\nu|(B_{7/8}).$$

Fixing ε and letting $\delta \downarrow 0$ first gives $\limsup_{\delta\downarrow 0}[\delta^2\log(1/\delta)]^{-1}\int_{B_\delta}|v|\,\mathrm{d}x \le C|\nu|(B_{2\varepsilon})$. Since $|\nu|(\{0\})=0$, letting $\varepsilon\downarrow 0$ yields

$$\frac{1}{\delta^2\log(1/\delta)}\int_{B_\delta}|v|\,\mathrm{d}x \to 0 \qquad \text{as } \delta\downarrow 0.$$

Moreover,

$$\frac{1}{\delta^2\log(1/\delta)}\int_{B_\delta}\Gamma(x)\,\mathrm{d}x \to \frac{1}{2}, \qquad \frac{1}{\delta^2\log(1/\delta)}\int_{B_\delta}|H|\,\mathrm{d}x \to 0, \quad \text{as } \delta\to 0.$$

Again, $a<0$ contradicts $u\ge 0$. Therefore, $a\ge 0$, and (3.5) becomes

$$\mathscr{L}_s u + cu = \mu + a\delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

□

4. Proofs of the maximum principles

The following argument is the mixed-operator analogue of [40, Theorems 1 and 2].

Proof of Theorem 1.3. Set $\Lambda = \|c^+\|_{L^\infty(B_r(x_0))}$. Since $\Lambda - c \ge 0$ and $u\ge 0$ in $B_r(x_0)$,

$$\mathscr{L}_s u + \Lambda u = (\mathscr{L}_s u + cu) + (\Lambda - c)u \ge 0 \quad \text{in } \mathcal{D}'(B_r(x_0)\setminus\{x_0\}).$$

The corresponding ball-version of Theorem 1.2, applied with the constant coefficient Λ, then gives

$$\mathscr{L}_s u + \Lambda u \ge 0 \quad \text{in } \mathcal{D}'(B_r(x_0)).$$

We first assume that u is smooth in the ball. Let $x_*\in\overline{B_{r/2}(x_0)}$ be a minimum point. If $u(x_*)\ge m$ there is nothing to prove. Otherwise x_* is an interior point and $-\Delta u(x_*)\le 0$. Since $u(x_*)\le u$ in $B_{r/2}(x_0)$, $u\ge m$ on $B_r(x_0)\setminus B_{r/2}(x_0)$, and $u\ge 0$ in $\mathbb{R}^n$, we obtain

$$(-\Delta)^s u(x_*) \le C_{n,s}I_r(x_*)u(x_*) + C_{n,s}J_r(x_*)\big(u(x_*)-m\big),$$

where

$$I_r(x) = \int_{\mathbb{R}^n\setminus B_r(x_0)}\frac{\mathrm{d}y}{|x-y|^{n+2s}}, \qquad J_r(x) = \int_{B_r(x_0)\setminus B_{r/2}(x_0)}\frac{\mathrm{d}y}{|x-y|^{n+2s}}.$$

For $x\in B_{r/2}(x_0)$,

$$I_r(x)\le C_1 r^{-2s}, \qquad J_r(x)\ge C_2 r^{-2s},$$

with $C_1, C_2>0$ depending only on n,s. Evaluating $\mathscr{L}_s u+\Lambda u\ge 0$ at x_* therefore gives

$$\frac{u(x_*)}{m} \ge \frac{C_{n,s}C_2}{C_{n,s}(C_1+C_2)+\Lambda r^{2s}} = \alpha(n,s,\Lambda r^{2s}) > 0.$$

This proves the estimate for smooth supersolutions.

For the distributional case, Lemma A.2 gives

$$(\mathscr{L}_s+\Lambda)u^\delta \ge 0 \quad \text{in } B_{r-\delta}(x_0).$$

Moreover

$$u^\delta \ge m \quad \text{in } B_{r-\delta}(x_0)\setminus B_{r/2+\delta}(x_0).$$

Repeating the preceding minimum argument with the radii $r/2+\delta$ and $r-\delta$ gives, for $0<\delta<r/8$,

$$u^\delta \geq \alpha_0 m \quad \text{in } B_{r/2+\delta}(x_0),$$

where

$$\alpha_0 = \alpha_0(n, s, \Lambda r^{2s}) > 0$$

is independent of δ. Letting $\delta \downarrow 0$ and using $u^\delta \to u$ in L^1_{loc} proves the assertion a.e., compare [40, Theorem 1]. □

Proof of Theorem 1.4. Again put $\Lambda = \|c^+\|_{L^\infty(B_r(x_0))}$. Since $u \geq 0$ in $B_r(x_0) \subset H$,

$$\mathscr{L}_s u + \Lambda u = (\mathscr{L}_s u + cu) + (\Lambda - c)u \geq 0 \quad \text{in } \mathcal{D}'(B_r(x_0) \setminus \{x_0\}).$$

The corresponding ball-version of Theorem 1.2, applied with coefficient Λ, gives

$$\mathscr{L}_s u + \Lambda u \geq 0 \quad \text{in } \mathcal{D}'(B_r(x_0)).$$

Assume first that u is smooth. For $x, y \in H$ set

$$K_H(x,y) = \frac{1}{|x-y|^{n+2s}} - \frac{1}{|x-\widetilde{y}|^{n+2s}} > 0.$$

Using antisymmetry and pairing H with its reflection,

$$(-\Delta)^s u(x) = C_{n,s}\,\mathrm{PV}\int_H (u(x)-u(y))K_H(x,y)\,\mathrm{d}y + 2C_{n,s}u(x)\int_H \frac{\mathrm{d}y}{|x-\widetilde{y}|^{n+2s}}. \tag{4.1}$$

Let x_* be a minimum of u on $\overline{B_{r/2}(x_0)}$. If $u(x_*) < m$, then x_* is interior and $-\Delta u(x_*) \leq 0$. Splitting the first integral in (4.1) into $B_{r/2}(x_0)$, the annulus $B_r(x_0)\backslash B_{r/2}(x_0)$, and $H \setminus B_r(x_0)$ gives

$$(-\Delta)^s u(x_*) \leq C_{n,s}\widehat{I}_r(x_*)u(x_*) + C_{n,s}\widehat{J}_r(x_*)\big(u(x_*) - m\big),$$

where

$$\widehat{I}_r(x) = \int_{H\backslash B_r(x_0)} K_H(x,y)\,\mathrm{d}y + 2\int_H \frac{\mathrm{d}y}{|x-\widetilde{y}|^{n+2s}},$$

$$\widehat{J}_r(x) = \int_{B_r(x_0)\backslash B_{r/2}(x_0)} K_H(x,y)\,\mathrm{d}y.$$

Since $B_r(x_0) \subset H$, every $x \in B_{r/2}(x_0)$ satisfies $x_1 \geq r/2$. Hence $\widehat{I}_r(x) \leq C_3 r^{-2s}$. Moreover $\widehat{J}_r(x) \geq C_4 r^{-2s}$. Indeed, set

$$E = B_{r/8}(x_0 + 3re_1/4) \subset B_r(x_0) \setminus B_{r/2}(x_0).$$

For $x \in B_{r/2}(x_0)$ and $y \in E$,

$$|x-y| \leq \frac{11}{8}r, \qquad |x-\widetilde{y}| \geq x_1 + |y_1| \geq \frac{17}{8}r,$$

because $B_r(x_0) \subset H$ implies $(x_0)_1 \geq r$. Consequently

$$K_H(x,y) \geq \left[\left(\frac{8}{11}\right)^{n+2s} - \left(\frac{8}{17}\right)^{n+2s}\right] r^{-n-2s},$$

and integration over E gives the claimed lower bound. Evaluating $\mathscr{L}_s u + \Lambda u \geq 0$ at x_* gives

$$\frac{u(x_*)}{m} \geq \frac{C_{n,s}C_4}{C_{n,s}(C_3 + C_4) + \Lambda r^{2s}} = \alpha(n, s, \Lambda r^{2s}) > 0.$$

For the distributional case choose j radial and radially nonincreasing. Lemma A.2 gives

$$(\mathscr{L}_s + \Lambda)u^\delta \geq 0 \quad \text{in } B_{r-\delta}(x_0),$$

and antisymmetry gives, for $x \in H$,

$$u^\delta(x) = \int_H u(y)\big(j_\delta(x - y) - j_\delta(x - \widetilde{y})\big)\,\mathrm{d}y \geq 0,$$

since $|x - y| \leq |x - \widetilde{y}|$. Also

$$u^\delta \geq m \quad \text{in } B_{r-\delta}(x_0) \setminus B_{r/2+\delta}(x_0).$$

The preceding estimate, with these perturbed radii, is uniform for $0 < \delta < r/8$. Letting $\delta \downarrow 0$ proves the assertion a.e., compare [40, Theorem 2]. ☐

Appendix A. Auxiliary properties of the mixed operator

Proposition A.1 (Maximum of subsolutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $u, v \in \mathcal{L}^{2s}(\mathbb{R}^n)$, $f, g \in L^1_{\mathrm{loc}}(\Omega)$, and $c \in L^\infty(\Omega)$. If*

$$\mathscr{L}_s u + cu \leq f, \qquad \mathscr{L}_s v + cv \leq g \quad \textit{in } \mathcal{D}'(\Omega),$$

then, for $w = \max\{u, v\}$,

$$\mathscr{L}_s w + cw \leq f\chi_{\{u>v\}} + g\chi_{\{u<v\}} + \max\{f, g\}\chi_{\{u=v\}} \quad \textit{in } \mathcal{D}'(\Omega). \tag{A.1}$$

Proof. Since $|w| \leq |u| + |v|$, we have $w \in \mathcal{L}^{2s}(\mathbb{R}^n)$. Put $F = f - cu$ and $G = g - cv$, mollify the two inequalities, and let $M_\varepsilon(a, b) = b + \theta_\varepsilon(a - b)$, where θ_ε is a smooth convex approximation of t^+ and $p_\varepsilon = \theta'_\varepsilon(u^\delta - v^\delta) \in [0, 1]$. The local chain rule and the supporting-plane inequality for the fractional term give

$$\mathscr{L}_s M_\varepsilon(u^\delta, v^\delta) \leq p_\varepsilon \mathscr{L}_s u^\delta + (1 - p_\varepsilon)\mathscr{L}_s v^\delta \leq p_\varepsilon F^\delta + (1 - p_\varepsilon)G^\delta.$$

Letting $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ gives the standard Kato limit (compare [40, Lemma 2]): on $\{u > v\}$ and $\{u < v\}$ the weights converge to 1 and 0, respectively, while on $\{u = v\}$ every limiting weight lies in $[0, 1]$. Hence

$$\mathscr{L}_s w \leq F\chi_{\{u>v\}} + G\chi_{\{u<v\}} + \max\{F, G\}\chi_{\{u=v\}}.$$

Adding cw gives (A.1), since $w = u$ on $\{u > v\}$, $w = v$ on $\{u < v\}$, and $u = v = w$ on $\{u = v\}$. ☐

Let $j \in C_c^\infty(B_1)$ be nonnegative and radial, with $\int j = 1$, and set

$$j_\delta(x) = \delta^{-n} j(x/\delta), \qquad w^\delta = w * j_\delta.$$

Lemma A.2 (Mollification)**.** *Let $w \in \mathcal{L}^{2s}(\mathbb{R}^n)$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. If*

$$\mathscr{L}_s w \le f \quad \text{in } \mathcal{D}'(\Omega),$$

then

$$\mathscr{L}_s w^\delta \le f^\delta \quad \text{in } \mathcal{D}'(\Omega^\delta), \qquad \Omega^\delta = \{x \in \mathbb{R}^n \mid B_\delta(x) \subset \Omega\}.$$

The same statement holds with $\mathscr{L}_s$ replaced by $\mathscr{L}_s + c_0$ for any constant $c_0 \in \mathbb{R}$, and the analogous assertions hold with all inequalities reversed.

Proof. For $0 \le \varphi \in C_c^\infty(\Omega^\delta)$,

$$\langle \mathscr{L}_s w^\delta - f^\delta, \varphi \rangle = \int_{B_\delta} j_\delta(z) \langle \mathscr{L}_s w - f, \varphi(\cdot + z) \rangle \, \mathrm{d}z \le 0.$$

The constant zero-order term commutes with convolution. For the purely fractional version, compare [40, Lemma 1]. □

Theorem A.3 (Interior smoothness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $w \in \mathcal{L}^{2s}(\mathbb{R}^n)$. If $\mathscr{L}_s w = 0$ in $\mathcal{D}'(\Omega)$, then $w \in C^\infty(\Omega)$.*

Proof. Fix smooth subdomains $U \Subset V \Subset \Omega$ and choose $\chi \in C_c^\infty(\Omega)$ with $\chi = 1$ in a neighborhood of $\overline{V}$. Since $\chi w \in L^1_c(\mathbb{R}^n)$, one has $\chi w \in H^{-N}(\mathbb{R}^n)$ for every $N > n/2$. For $x \in V$, in the distributional sense,

$$(-\Delta)^s w = (-\Delta)^s(\chi w) - g_\chi, \qquad g_\chi(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{(1-\chi(y))w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y. \tag{A.2}$$

For every multi-index γ and every $x \in V$,

$$\partial_x^\gamma g_\chi(x) = C_{n,s} \int_{\mathbb{R}^n} (1-\chi(y)) w(y) \partial_x^\gamma |x-y|^{-n-2s} \, \mathrm{d}y.$$

The distance between V and $\mathrm{supp}(1-\chi)$ is positive, and the tail assumption on w makes the last integral absolutely convergent after any number of x-derivatives. Hence $g_\chi \in C^\infty(V)$.

Assume now that $w \in H^t_{\mathrm{loc}}(\Omega)$ for some $t \in \mathbb{R}$. Since $\chi \in C_c^\infty(\Omega)$, we have $\chi w \in H^t(\mathbb{R}^n)$. The Fourier multiplier representation of $(-\Delta)^s$ gives

$$\|(-\Delta)^s(\chi w)\|_{H^{t-2s}(\mathbb{R}^n)} \le C \|\chi w\|_{H^t(\mathbb{R}^n)}. \tag{A.3}$$

Since $g_\chi \in C^\infty(V)$, (A.3) and (A.2) imply $(-\Delta)^s w \in H^{t-2s}_{\mathrm{loc}}(V)$. Using $\mathscr{L}_s w = 0$ in $\mathcal{D}'(\Omega)$, we obtain $-\Delta w = -(-\Delta)^s w \in H^{t-2s}_{\mathrm{loc}}(V)$. The local elliptic regularity theorem for the Laplacian yields $w \in H^{t+2-2s}_{\mathrm{loc}}(V)$. Since $U \Subset V \Subset \Omega$ are arbitrary, we have proved the bootstrap implication

$$w \in H^t_{\mathrm{loc}}(\Omega) \implies w \in H^{t+2-2s}_{\mathrm{loc}}(\Omega). \tag{A.4}$$

The argument at the beginning of the proof, applied to arbitrary compactly supported cutoffs, shows that

$$w \in H^{-N}_{\mathrm{loc}}(\Omega) \qquad \text{for every } N > \frac{n}{2}.$$

Iterating (A.4), we obtain

$$w \in H^{-N+k(2-2s)}_{\mathrm{loc}}(\Omega) \qquad \text{for every } k \in \mathbb{N}.$$

The Sobolev embedding theorem then gives $w \in C^m(U)$ for every $m \in \mathbb{N}$. Since $U \Subset \Omega$ are arbitrary, it follows that $w \in C^\infty(\Omega)$. ☐

Appendix B. Homogeneous Distributions

Proposition B.1 (Fourier transforms of radial homogeneous distributions)**.** *Under the normalization* (1.1)*, the following statements hold.*

(1) *For* $0 < \alpha < n$*,*

$$\mathcal{F}^{-1}(|\xi|^{-\alpha})(x) = \kappa_{n,\alpha}|x|^{\alpha-n}. \tag{B.1}$$

The family $|\xi|^{-\alpha}$*, initially defined as a tempered distribution for* $\Re\alpha < n$*, admits a meromorphic continuation to* $\alpha \in \mathbb{C}$*, with possible poles at* $\alpha = n, n+2, n+4, \cdots$*. The identity* (B.1) *extends accordingly by meromorphic continuation. Away from the origin, this continuation is represented by* $\kappa_{n,\alpha}|x|^{\alpha-n}$*. In particular, if* $\alpha = -2k$*,* $k \in \mathbb{N}_0$*, then*

$$\mathcal{F}^{-1}(|\xi|^{-\alpha}) = (-\Delta)^k\delta_0.$$

(2) *Define*

$$\left\langle \mathrm{Fp}\,|\xi|^{-n}, \varphi \right\rangle = \int_{|\xi|<1} \frac{\varphi(\xi)-\varphi(0)}{|\xi|^n}\,\mathrm{d}\xi + \int_{|\xi|>1} \frac{\varphi(\xi)}{|\xi|^n}\,\mathrm{d}\xi, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Then

$$\mathcal{F}^{-1}\big(\mathrm{Fp}\,|\xi|^{-n}\big) = c_n \log\frac{1}{|x|} + c_n\left(\log 2 + \frac{1}{2}\psi\Big(\frac{n}{2}\Big) - \frac{\gamma}{2}\right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{B.2}$$

where

$$c_n = \frac{2^{1-n}}{\pi^{n/2}\Gamma(n/2)},$$

γ *is Euler's constant, and* $\psi = \Gamma'/\Gamma$*.*

Proof. For $0 < \alpha < n$,

$$|\xi|^{-\alpha} = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}e^{-t|\xi|^2}\,\mathrm{d}t,$$

and

$$\mathcal{F}^{-1}(e^{-t|\xi|^2})(x) = (4\pi t)^{-n/2}e^{-|x|^2/(4t)}.$$

Hence, for $x \neq 0$,

$$\begin{aligned}\mathcal{F}^{-1}(|\xi|^{-\alpha})(x) &= \frac{(4\pi)^{-n/2}}{\Gamma(\alpha/2)}\int_0^\infty t^{(\alpha-n)/2-1}e^{-|x|^2/(4t)}\,\mathrm{d}t\\ &= \frac{\Gamma\big(\frac{n-\alpha}{2}\big)}{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}|x|^{\alpha-n}.\end{aligned}$$

This proves (B.1) for $0 < \alpha < n$.

We next construct the meromorphic continuation. Fix $N \geq 0$, and let $P_N\varphi$ denote the Taylor polynomial of φ at the origin of degree N. Initially, for $\Re\alpha < n$,

$$\int_{\mathbb{R}^n} |\xi|^{-\alpha}\varphi(\xi)\,\mathrm{d}\xi = \int_{|\xi|<1} |\xi|^{-\alpha}\big(\varphi(\xi) - P_N\varphi(\xi)\big)\,\mathrm{d}\xi + \int_{|\xi|>1} |\xi|^{-\alpha}\varphi(\xi)\,\mathrm{d}\xi + \sum_{|\rho|\leq N} \frac{\partial^\rho\varphi(0)}{\rho!} \int_{|\xi|<1} |\xi|^{-\alpha}\xi^\rho\,\mathrm{d}\xi.$$

For $\Re\alpha < n + |\rho|$, polar coordinates give

$$\int_{|\xi|<1} |\xi|^{-\alpha}\xi^\rho\,\mathrm{d}\xi = \frac{1}{n + |\rho| - \alpha} \int_{\mathbb{S}^{n-1}} \theta^\rho\,\mathrm{d}\theta.$$

The first integral above, containing $\varphi - P_N\varphi$, is holomorphic for $\Re\alpha < n + N + 1$, since

$$\varphi(\xi) - P_N\varphi(\xi) = O(|\xi|^{N+1}) \qquad \text{as } \xi \to 0.$$

The integral over $|\xi| > 1$ is entire in α, because φ is a Schwartz function. It follows that, for $\Re\alpha < n + N + 1$, the continuation is given by

$$\langle T_\alpha, \varphi\rangle = \int_{|\xi|<1} |\xi|^{-\alpha}\big(\varphi(\xi) - P_N\varphi(\xi)\big)\,\mathrm{d}\xi + \int_{|\xi|>1} |\xi|^{-\alpha}\varphi(\xi)\,\mathrm{d}\xi + \sum_{|\rho|\leq N} \frac{\partial^\rho\varphi(0)}{\rho!} \frac{1}{n + |\rho| - \alpha} \int_{\mathbb{S}^{n-1}} \theta^\rho\,\mathrm{d}\theta.$$

Thus the only possible poles in this half-plane are $n, n+1, \ldots, n+N$. If $|\rho|$ is odd, then $\int_{\mathbb{S}^{n-1}} \theta^\rho\,\mathrm{d}\theta = 0$, so the possible poles reduce to $n, n+2, n+4, \cdots$. The continuations corresponding to different values of N agree on their common domains and hence define an $\mathcal{S}'$-valued meromorphic family T_α on $\mathbb{C}$.

For $0 < \alpha < n$, the computation above gives $\mathcal{F}^{-1}T_\alpha = \kappa_{n,\alpha}|x|^{\alpha-n}$. Since the inverse Fourier transform is continuous on $\mathcal{S}'$, uniqueness of meromorphic continuation extends this identity to all noncritical values of α. If $\alpha = -2k$, then $T_{-2k} = |\xi|^{2k}$, and therefore $\mathcal{F}^{-1}T_{-2k} = (-\Delta)^k\delta_0$.

We now consider the first critical value $\alpha = n$. For $\varepsilon > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, splitting at $|\xi| = 1$ and subtracting $\varphi(0)$ on the unit ball gives

$$|\xi|^{-n+\varepsilon} = \frac{\omega_{n-1}}{\varepsilon}\delta_0 + \operatorname{Fp}|\xi|^{-n} + \varepsilon R_\varepsilon \qquad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{B.3}$$

where

$$\langle R_\varepsilon, \varphi\rangle = \int_{|\xi|<1} \frac{\varphi(\xi) - \varphi(0)}{|\xi|^n} \frac{|\xi|^\varepsilon - 1}{\varepsilon}\,\mathrm{d}\xi + \int_{|\xi|>1} \frac{\varphi(\xi)}{|\xi|^n} \frac{|\xi|^\varepsilon - 1}{\varepsilon}\,\mathrm{d}\xi. \tag{B.4}$$

Indeed,

$$\int_{|\xi|<1} |\xi|^{-n+\varepsilon}\,\mathrm{d}\xi = \omega_{n-1}\int_0^1 r^{-1+\varepsilon}\,\mathrm{d}r = \frac{\omega_{n-1}}{\varepsilon}.$$

Fix $0<\varepsilon\le\varepsilon_0$. The mean-value theorem gives

$$\left|\frac{r^\varepsilon-1}{\varepsilon}\right|\le|\log r|,\quad 0<r<1,\quad\text{and}\quad\left|\frac{r^\varepsilon-1}{\varepsilon}\right|\le r^{\varepsilon_0}\log r,\quad r>1.$$

Since

$$|\varphi(\xi)-\varphi(0)|\le C|\xi|$$

near the origin, the first integral in (B.4) is bounded by a constant multiple of

$$\int_{|\xi|<1}|\xi|^{1-n}|\log|\xi||\,\mathrm{d}\xi<\infty.$$

The second integral is bounded by a fixed Schwartz seminorm of φ. Hence R_ε remains bounded in $\mathcal{S}'$.

Taking inverse Fourier transforms in (B.3) and using $\mathcal{F}^{-1}\delta_0=(2\pi)^{-n}$, we obtain

$$\mathcal{F}^{-1}(|\xi|^{-n+\varepsilon})=\frac{c_n}{\varepsilon}+\mathcal{F}^{-1}\big(\mathrm{Fp}\,|\xi|^{-n}\big)+O_{\mathcal{S}'}(\varepsilon),\tag{B.5}$$

where

$$c_n=\omega_{n-1}(2\pi)^{-n}=\frac{2^{1-n}}{\pi^{n/2}\Gamma(n/2)}.$$

On the other hand, (B.1) with $\alpha=n-\varepsilon$ gives

$$\mathcal{F}^{-1}(|\xi|^{-n+\varepsilon})(x)=\frac{\Gamma(\varepsilon/2)}{2^{n-\varepsilon}\pi^{n/2}\Gamma\big(\frac{n-\varepsilon}{2}\big)}|x|^{-\varepsilon}.$$

We use

$$\Gamma(\varepsilon/2)=\frac{2}{\varepsilon}-\gamma+O(\varepsilon),\qquad 2^\varepsilon=1+\varepsilon\log 2+O(\varepsilon^2),$$
$$\frac{1}{\Gamma\big(\frac{n-\varepsilon}{2}\big)}=\frac{1}{\Gamma(n/2)}\left(1+\frac{\varepsilon}{2}\psi\Big(\frac{n}{2}\Big)+O(\varepsilon^2)\right).$$

It remains to justify the expansion of $|x|^{-\varepsilon}$ in $\mathcal{S}'$. Choose $0<\varepsilon_0<n$. Taylor's formula gives, uniformly for $0<\varepsilon\le\varepsilon_0$,

$$\left|\frac{|x|^{-\varepsilon}-1+\varepsilon\log|x|}{\varepsilon^2}\right|\le C\begin{cases}|x|^{-\varepsilon_0}\big(1+|\log|x||^2\big), & |x|<1,\\ 1+|\log|x||^2, & |x|\ge 1.\end{cases}$$

The right-hand side is locally integrable and has at most logarithmic growth at infinity. Hence

$$|x|^{-\varepsilon}=1-\varepsilon\log|x|+O_{\mathcal{S}'}(\varepsilon^2).$$

Combining the preceding expansions yields

$$\mathcal{F}^{-1}(|\xi|^{-n+\varepsilon})=\frac{c_n}{\varepsilon}+c_n\log\frac{1}{|x|}+c_n\left(\log 2+\frac{1}{2}\psi\Big(\frac{n}{2}\Big)-\frac{\gamma}{2}\right)+O_{\mathcal{S}'}(\varepsilon).$$

Comparison with (B.5) gives (B.2). ∎

Appendix C. Cutoff Multipliers and Dyadic Kernel Estimates

Lemma C.1 (Cutoff and symbol estimates)**.**

(1) *Suppose that $a \in C^\infty(\mathbb{R}^n)$ vanishes near the origin and that, for some $\mu \in \mathbb{R}$ and every multi-index ρ,*

$$|\partial_\xi^\rho a(\xi)| \le C_\rho |\xi|^{-\mu-|\rho|} \qquad \text{for any } |\xi| \ge 1. \tag{C.1}$$

Then $K = \mathcal{F}^{-1}a$ is smooth on $\mathbb{R}^n \setminus \{0\}$, is rapidly decreasing at infinity, and for every multi-index γ and $0 < |x| \le 1$,

$$|\partial_x^\gamma K(x)| \le \begin{cases} C_\gamma |x|^{\mu-n-|\gamma|}, & \mu < n + |\gamma|, \\ C_\gamma (1 + |\log|x||), & \mu = n + |\gamma|, \\ C_\gamma, & \mu > n + |\gamma|. \end{cases} \tag{C.2}$$

If, in addition, a is radial, $\mu > n$, and

$$0 < \beta < \min\{\mu - n, 2\}, \tag{C.3}$$

then

$$K(x) = K(0) + o(|x|^\beta), \qquad \nabla K(x) = o(|x|^{\beta-1}), \qquad \text{as } |x| \to 0. \tag{C.4}$$

(2) *Suppose that b is compactly supported, smooth on $\mathbb{R}^n \setminus \{0\}$, and that, for some $\nu > -n$ and every multi-index ρ,*

$$|\partial_\xi^\rho b(\xi)| \le C_\rho |\xi|^{\nu-|\rho|} \qquad \text{for any } 0 < |\xi| \le 2.$$

Then, for every multi-index γ and $|x| \ge 1$,

$$|\partial_x^\gamma \mathcal{F}^{-1} b(x)| \le C_\gamma |x|^{-n-\nu-|\gamma|}. \tag{C.5}$$

Proof. Choose $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, supported in $\{\frac{1}{2} \le |\xi| \le 2\}$, such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1 \qquad \text{for any } \xi \ne 0.$$

High frequencies. Because a vanishes near the origin, after a harmless shift of the index,

$$a = \sum_{j=0}^\infty a_j \quad \text{in } \mathcal{S}', \qquad a_j(\xi) = \psi(2^{-j}\xi)a(\xi).$$

Each a_j is supported where $|\xi| \asymp 2^j$. For a multi-index γ,

$$K_{j,\gamma}(x) = \partial_x^\gamma \mathcal{F}^{-1} a_j(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} (i\xi)^\gamma a_j(\xi)\, \mathrm{d}\xi.$$

After $\xi = 2^j\zeta$,

$$K_{j,\gamma}(x) = 2^{j(n+|\gamma|-\mu)} \frac{1}{(2\pi)^n} \int e^{i2^j x\cdot\zeta} A_{j,\gamma}(\zeta)\, \mathrm{d}\zeta,$$

where

$$A_{j,\gamma}(\zeta) = 2^{j\mu} (i\zeta)^\gamma \psi(\zeta) a(2^j\zeta).$$

The amplitudes $A_{j,\gamma}$ are supported in one fixed annulus and are uniformly bounded in every C^M-norm. Indeed, every derivative of $a(2^j\zeta)$ contributes a factor 2^j, which is canceled by the decay in (C.1).

For an integer $L \geq 0$,

$$(1-\Delta_\zeta)^L e^{i2^j x\cdot\zeta} = (1+2^{2j}|x|^2)^L e^{i2^j x\cdot\zeta}.$$

Since the amplitude is compactly supported, integration by parts gives

$$|K_{j,\gamma}(x)| \leq C_{N,\gamma} 2^{j(n+|\gamma|-\mu)}(1+2^j|x|)^{-N} \tag{C.6}$$

for every $N \geq 0$. This is first obtained for even integers $N = 2L$, and the remaining values follow by weakening the estimate.

For every compact set $E \Subset \mathbb{R}^n\setminus\{0\}$, choosing $N > n+|\gamma|-\mu$ makes $\sum_j K_{j,\gamma}$ uniformly convergent on E. Hence the dyadic sum is smooth away from the origin. Since the partial sums converge to a in $\mathcal{S}'$, this smooth function represents $\mathcal{F}^{-1}a$ there.

Let $0 < |x| \leq 1$, choose $J \geq 0$ such that $2^J|x| \leq 1 < 2^{J+1}|x|$, and put $A = n+|\gamma|-\mu$. For $j \leq J$,

$$\sum_{j=0}^{J} |K_{j,\gamma}(x)| \leq C\sum_{j=0}^{J} 2^{jA} \leq \begin{cases} C|x|^{-A}, & A>0,\\ C(1+|\log|x||), & A=0,\\ C, & A<0. \end{cases}$$

For $j > J$, choose $N > \max\{A, 0\}$. Then

$$\begin{aligned}\sum_{j>J} |K_{j,\gamma}(x)| &\leq C|x|^{-N}\sum_{j>J} 2^{j(A-N)}\\ &\leq C|x|^{-N}2^{J(A-N)} \leq C|x|^{-A}.\end{aligned}$$

If $A \leq 0$, the last expression is bounded. This proves (C.2).

For $|x| \geq 1$, choose $N > M + \max\{A, 0\}$ in (C.6). Then

$$\sum_{j=0}^{\infty} |K_{j,\gamma}(x)| \leq C|x|^{-N}\sum_{j=0}^{\infty} 2^{j(A-N)} \leq C_{M,\gamma}|x|^{-M}.$$

Thus K and all its derivatives are rapidly decreasing at infinity.

Assume now that a is radial, $\mu > n$, and (C.3) holds. Then $a \in L^1$, and

$$K(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}a(\xi)\,\mathrm{d}\xi.$$

Choose $\beta < \beta' < \min\{\mu-n, 2\}$. Then $\int |\xi|^{\beta'}|a(\xi)|\,\mathrm{d}\xi < \infty$. If $0 < \beta' \leq 1$, use $|e^{it}-1| \leq C|t|^{\beta'}$. If $1 < \beta' < 2$, radiality implies that a is even and $\int \xi a(\xi)\,\mathrm{d}\xi = 0$, use $|e^{it}-1-it| \leq C|t|^{\beta'}$. In both cases,

$$|K(x)-K(0)| \leq C|x|^{\beta'}\int |\xi|^{\beta'}|a(\xi)|\,\mathrm{d}\xi = o(|x|^{\beta}).$$

For the gradient, suppose first that $0 < \beta < 1$. If $\mu < n+1$, (C.2) gives

$$|\nabla K(x)| \leq C|x|^{\mu-n-1} = o(|x|^{\beta-1}),$$

because $\beta < \mu - n$. If $\mu = n+1$, the logarithmic bound has the same conclusion. If $\mu > n+1$, boundedness suffices. If $\beta = 1$, then $\mu > n+1$ and $\xi a \in L^1$. Hence ∇K is continuous and $\nabla K(0) = 0$ by radiality. If $1 < \beta < 2$, then

$$\begin{aligned} |\nabla K(x) - \nabla K(0)| &\le C \int |\xi|\, |e^{ix\cdot\xi} - 1|\, |a(\xi)|\, \mathrm{d}\xi \\ &\le C|x|^{\beta'-1} \int |\xi|^{\beta'} |a(\xi)|\, \mathrm{d}\xi = o(|x|^{\beta-1}). \end{aligned}$$

This proves (C.4).

Low frequencies. Since $\nu > -n$, the function b is locally integrable near the origin. Decompose

$$b = b_* + \sum_{j=0}^{\infty} b_j, \qquad b_j(\xi) = \psi(2^j\xi) b(\xi),$$

where $b_* \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ collects the finitely many annuli away from the origin. The inverse Fourier transform of b_* is Schwartz. The support of b_j lies where $|\xi| \asymp 2^{-j}$. Rescaling $\xi = 2^{-j}\zeta$ and repeating the previous integration-by-parts argument gives

$$|\partial_x^\gamma \mathcal{F}^{-1} b_j(x)| \le C_{N,\gamma} 2^{-j(n+\nu+|\gamma|)} (1 + 2^{-j}|x|)^{-N}.$$

Let $|x| \ge 1$, choose $J \ge 0$ such that $2^J \le |x| < 2^{J+1}$, and set $B = n + \nu + |\gamma| > 0$. For $j \ge J$,

$$\sum_{j \ge J} |\partial_x^\gamma \mathcal{F}^{-1} b_j(x)| \le C \sum_{j\ge J} 2^{-jB} \le C|x|^{-B}.$$

For $j < J$, choose $N > B$. Then

$$\begin{aligned} \sum_{j<J} |\partial_x^\gamma \mathcal{F}^{-1} b_j(x)| &\le C|x|^{-N} \sum_{j<J} 2^{j(N-B)} \\ &\le C|x|^{-N} 2^{J(N-B)} \le C|x|^{-B}. \end{aligned}$$

Adding the Schwartz contribution from b_* proves (C.5). ☐

Corollary C.2 (Effect of a frequency cutoff)**.** *Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be radial and equal to 1 near the origin, and set $\eta = 1 - \chi$.*

(1) *If $\alpha > 0$ and $\alpha \notin \{n, n+2, n+4, \dots\}$, then*

$$\mathcal{F}^{-1}(\eta|\xi|^{-\alpha})(x) = \kappa_{n,\alpha}|x|^{\alpha-n} + H_\alpha(x), \tag{C.7}$$

where H_α is smooth and radial near the origin. At $\alpha = n$,

$$\mathcal{F}^{-1}\big(\eta\, \mathrm{Fp}\, |\xi|^{-n}\big)(x) = c_n \log\frac{1}{|x|} + H_n(x), \tag{C.8}$$

where H_n is smooth and radial near the origin.

(2) *If $\alpha < n$ and $\alpha \notin \{0, -2, -4, \dots\}$, then, for every $N > 0$,*

$$\mathcal{F}^{-1}(\chi|\xi|^{-\alpha})(x) = \kappa_{n,\alpha}|x|^{\alpha-n} + O(|x|^{-N}), \qquad |x| \to \infty. \tag{C.9}$$

If $\alpha \in \{0, -2, -4, \dots\}$, then $\mathcal{F}^{-1}(\chi|\xi|^{-\alpha})$ is a Schwartz function.

Proof. Let T_α denote the homogeneous continuation of $|\xi|^{-\alpha}$ given by Proposition B.1. For the first assertion, since $\eta T_\alpha = T_\alpha - \chi T_\alpha$ and χT_α is a compactly supported distribution, its inverse Fourier transform is smooth. Proposition B.1 therefore gives (C.7). The smooth remainder is radial because both T_α and χ are radial.

At $\alpha = n$, the same argument applied to $\operatorname{Fp}|\xi|^{-n}$ gives (C.8).

For the second assertion, assume that $\alpha < n$ and $\alpha \notin \{0, -2, -4, \ldots\}$. In this range T_α agrees with the locally integrable function $|\xi|^{-\alpha}$. Since $\chi T_\alpha = T_\alpha - \eta T_\alpha$, Proposition B.1 gives $\mathcal{F}^{-1}T_\alpha(x) = \kappa_{n,\alpha}|x|^{\alpha-n}$ for $x \neq 0$. Moreover, $\eta T_\alpha = \eta|\xi|^{-\alpha}$ is smooth and satisfies the high-frequency symbol estimates in Lemma C.1, with $\mu = \alpha$. Hence $\mathcal{F}^{-1}(\eta T_\alpha)$ is rapidly decreasing at infinity, which proves (C.9).

Finally, if $\alpha = -2k$, $k \in \mathbb{N}_0$, then $\chi|\xi|^{-\alpha} = \chi|\xi|^{2k} \in C_c^\infty(\mathbb{R}^n)$, so its inverse Fourier transform is Schwartz. □

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