SURVEY OF THE ABREU EQUATION AND GUILLEMIN BOUNDARY PROBLEMS

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Abstract. This article provides a self-contained survey of the origin and development of the Abreu equation with Guillemin boundary conditions. We begin with the toric Kähler setting and Guillemin's construction of symplectic potentials, followed by Abreu's scalar curvature formula and the resulting fourth-order PDE. A central part of the survey is devoted to Donaldson's program on toric surfaces, including his variational formulation, stability notions, interior estimates for the Abreu equation, the M-condition, and existence theorems for constant scalar curvature metrics under toric K-stability assumptions. We then turn to the Monge-Ampère equation with Guillemin boundary data, discussing the pioneering work of Rubin, Huang's resolution of the polygonal two-dimensional case, the higher-dimensional theory of Huang-Shen, and recent Schauder-type boundary estimates by Bayrami-Seyyedali-Talebi. Finally, we discuss further developments in the PDE theory of the Abreu equation itself and outline its connections with affine differential geometry and the Monge-Ampère theory of Trudinger and Wang.

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Date: December 7, 2025.

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1. Introduction

The search for canonical metrics in Kähler geometry, such as Kähler-Einstein or constant scalar curvature Kähler (cscK) metrics, is tightly intertwined with questions of algebro-geometric stability. Following Calabi's reduction of the Calabi conjecture to a complex Monge-Ampère equation, Yau [Yau] solved this equation, proving the existence of Kähler-Einstein metrics on compact Kähler manifolds with $c_1 < 0$ or $c_1 = 0$. For the positive first Chern class, Tian [Ti1] initiated a stability-based approach to Kähler-Einstein metrics on Fano manifolds, introducing stability conditions that anticipate modern K-stability. Calabi [Ca] introduced extremal Kähler metrics as critical points of the L^2-norm of the scalar curvature, and Mabuchi [Ma] defined the K-energy functional whose gradient flow is formally the Calabi flow. These developments led to the Yau-Tian-Donaldson (YTD) picture, relating the existence of canonical metrics to K-stability [Do1]. In the Fano case, the YTD conjecture for Kähler-Einstein metrics has now been resolved by the works of Tian [Ti2] and of Chen-Donaldson-Sun [CDS1, CDS2, CDS3].

A major recent advance in this direction is the work of Chen and Cheng [CC1, CC2, CC3] on the cscK equation on arbitrary compact Kähler manifolds. In [CC1] they establish sharp *a priori* estimates, showing that higher-order bounds follow from a uniform C^0 estimate for the Kähler potential. In the companion paper [CC2] they prove that the properness of the Mabuchi K-energy with respect to the L^1 Mabuchi distance on the space of Kähler potentials implies the existence of a cscK metric, and that weak minimizers of the K-energy are automatically smooth cscK potentials, giving a powerful analytic version of the YTD picture. The subsequent work [CC3] treats general automorphism groups, showing how the existence theory must be formulated modulo the action of the identity component of $\mathrm{Aut}(X)$ and providing a refined properness criterion adapted to this symmetry. Together, these results place the existence theory for cscK metrics on a much more robust footing.

Despite this progress, the general cscK and extremal metric problem remains highly non-explicit: one rarely has a concrete analytic model for the relevant equations. Toric Kähler manifolds provide a striking exception. Here a large symmetry group allows one to encode the geometry in terms of convex analysis on a Delzant polytope, and the scalar curvature of a torus-invariant metric is expressed by an explicit, though highly nontrivial, fourth-order equation called the *Abreu equation*. This toric reduction makes it possible to study canonical metrics through a precise boundary value problem for a real PDE, and to isolate the delicate analytic issues that are otherwise obscured in the general setting.

On toric Kähler manifolds, the underlying symplectic geometry is described by the convexity results of Atiyah and of Guillemin-Sternberg [At, GS] and the classification theorem of Delzant [De]. Specifically, an effective Hamiltonian action of the real torus T^n on a compact symplectic manifold (X, ω) with moment map image $P \subset \mathbb{R}^n$ determines, and is determined by, a Delzant polytope P up to integral affine equivalence [De, Gu].

More precisely, let (X, ω) be a compact Kähler manifold of complex dimension n with an effective holomorphic Hamiltonian action of T^n. The corresponding moment map identifies X (up to equivariant symplectomorphism) with a symplectic toric manifold whose image is a Delzant polytope $P \subset \mathbb{R}^n$ [Gu, De]. Torus-invariant Kähler metrics in a fixed Kähler class correspond to convex functions u on P (the *symplectic potentials*), and the scalar curvature of the metric can be expressed as

$$S(u)(x) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}(x),$$

where u^{ij} denotes the inverse Hessian of u in the momentum coordinates [Ab]. This is the *Abreu equation* when one prescribes $S(u)$. In particular, toric cscK and extremal metrics correspond to solutions of the Abreu equation with $S(u)$ equal to a constant or an affine function on P.

The relevant convex functions u are not arbitrary; they must satisfy a specific asymptotic behaviour at the boundary of the polytope, the so-called *Guillemin boundary condition*. Roughly speaking, if

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\}, \qquad \ell_k(x) = \langle \nu_k, x \rangle + \lambda_k,$$

is a representation of P as an intersection of half-spaces, then the symplectic potential associated to any smooth toric metric has the form

$$u(x) = \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) + f(x),$$

with $f \in C^\infty(\overline{P})$ [Gu, Ab]. This condition is forced by the geometry near the toric divisors: the logarithmic terms encode the standard cone singularities of the moment map coordinates, while the smooth part f absorbs all global information. The Guillemin expansion will be a central player throughout this article.

From the geometric point of view, the most natural boundary value problem on a Delzant polytope P is therefore the following: given a smooth function $A \in C^\infty(\overline{P})$, find a strictly convex symplectic potential u such that

$$S(u) = A \quad \text{in } P, \qquad u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}). \tag{1.1}$$

We will refer to (1.1) as the *Guillemin boundary problem for the Abreu equation*. For suitable choices of A (for instance, $A \equiv \bar{S}$ or an affine function), this problem is equivalent to the existence of toric cscK or extremal metrics in a given Kähler class. In particular, Donaldson's program on toric surfaces [Do1, Do2, Do3, Do4] can be viewed as a deep existence and regularity theory for (1.1) coupled with a stability condition expressed directly on P.

From the PDE viewpoint, the Guillemin boundary problem for the Abreu equation exhibits two key analytic features. First, it is a fourth-order, fully nonlinear equation whose linearization has nontrivial kernel coming from holomorphic vector fields. Second, the boundary condition is highly singular: in the model Guillemin potential the Hessian degenerates like $1/\ell_k$ near a facet, while the domain P itself is only Lipschitz. These singularities are precisely what encode the compactification of the toric manifold, so they cannot be discarded by imposing, say, standard Dirichlet data on ∂P.

A powerful perspective, already implicit in Abreu's original work and developed further by Donaldson and others, is to rewrite the Abreu equation as a second-order system for the Hessian of u and its inverse. In this formulation, one of the equations is a real Monge-Ampère equation for u, while the other is a linear (or mildly nonlinear) divergence-type equation for the inverse Hessian. This leads naturally to the associated real Monge-Ampère equation

$$\det D^2 u = \frac{h(x)}{\prod_{k=1}^{N} \ell_k(x)} \quad \text{in } P, \tag{1.2}$$

coupled with the Guillemin boundary condition

$$u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}). \tag{1.3}$$

Here h is a positive smooth function on $\overline{P}$ determined by the scalar curvature data and the global geometry. The pair (1.2)-(1.3) is what is now called the *Guillemin boundary problem for the Monge-Ampère equation*. It may be viewed as a model problem that isolates the leading-order singular behaviour of solutions to (1.1).

The Monge-Ampère Guillemin problem has attracted substantial attention in recent years, both as a tool for understanding (1.1) and as an independent object in nonlinear analysis. In the polygonal case in dimension two, Rubin introduced this boundary value problem and established the existence of Alexandrov solutions with partial regularity [Ru]. Building on this, Huang obtained existence and higher regularity for smooth solutions on polygons [Hu]. More recently, Huang and Shen extended the theory to simple polytopes in arbitrary dimension, giving necessary and sufficient conditions for solvability and a refined analysis of the boundary singularities [HS]. Very recently, Bayrami-Aminlouee, Seyyedali, and Talebi established Schauder-type boundary estimates for the same singular Monge-Ampère equation on polygons with Guillemin boundary conditions, extending the regularity theory to Hölder right-hand sides and sharpening the understanding of corner behaviour [BST].

While the Monge-Ampère Guillemin problem captures the main singular features of toric Kähler metrics, the full geometric picture ultimately requires control of the Abreu equation with Guillemin boundary conditions. Here one finds a rich interaction between affine differential geometry, variational methods, and K-stability. Donaldson's work on toric surfaces [Do1, Do2, Do3, Do4] expresses the Mabuchi functional in terms of convex functions on P and relates its coercivity to an explicit polytope stability condition, leading to existence of cscK metrics under a uniform stability hypothesis. Subsequent work has developed interior estimates, regularity results, and generalized Abreu equations in higher dimensions, often working under the assumption that the solutions satisfy Guillemin-type boundary behavior. In particular, a substantial analytic toolkit, combining real and complex affine techniques, barrier constructions near facets, and blow-up analysis at vertices, has been built around the Guillemin boundary problem for the Abreu equation.

The primary aim of this survey is to give a structured and reasonably detailed account of the Guillemin boundary problem for the Abreu equation and its Monge-Ampère companion, viewed as a central model in the analytic study of canonical metrics on toric manifolds. More concretely, we will focus on:

- the geometric origin of Abreu's scalar curvature formula, the symplectic potential, and the Guillemin boundary condition, and how the toric extremal/cscK problem reduces to the Guillemin boundary problem (1.1);
- Donaldson's program on toric surfaces and its impact on the analysis of the Abreu equation with Guillemin boundary conditions, including the variational formulation and stability criteria on the polytope;
- the development of the Monge-Ampère equation with Guillemin boundary conditions (Rubin, Huang, Huang-Shen, Bayrami-Seyyedali-Talebi) and its role as a model problem for the singular geometry at the boundary of P;

- further PDE results for the Abreu equation and related fourth-order problems, and some open directions, placed in the broader context of Monge-Ampère and affine differential geometry.

In parallel, we will also formulate explicitly a Guillemin-type boundary value problem for the full Abreu equation itself; this will be taken up in a joint work of the author with Zhou [WZ].

The rest of the paper is organized as follows. Section 2 recalls basic facts on toric Kähler geometry, symplectic potentials, and the Guillemin expansion. In Section 3, we review Abreu's scalar curvature formula, the associated equation, the extremal condition in toric coordinates, and formulate the Guillemin boundary problem for the Abreu equation. Section 4 is devoted to Donaldson's program on toric surfaces and its variational/stability framework. Section 5 treats the Monge-Ampère equation with Guillemin boundary conditions and its recent developments. Further PDE results for the Abreu equation and related fourth-order problems, as well as their connections with K-stability, are collected in Section 6. Connections with affine differential geometry and Monge-Ampère theory, together with some discussions for future research, are discussed in Section 7.

2. Toric Kähler geometry and symplectic potentials

In this section we recall the basic dictionary between toric Kähler manifolds and convex geometry on their moment polytopes. We emphasize three complementary viewpoints:

- the symplectic description via Hamiltonian torus actions and Delzant polytopes;
- the complex description via invariant Kähler potentials on $(\mathbb{C}^*)^n$;
- the symplectic potentials and the Guillemin boundary condition on the polytope.

This material is classical, going back to Atiyah, Guillemin-Sternberg, Delzant and Guillemin [At, GS, De, Gu].

2.1. **Symplectic toric manifolds and Delzant polytopes.** Let (X, ω) be a compact connected symplectic manifold of dimension $2n$ equipped with an effective Hamiltonian action of the real torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Denote by $\mathfrak{t} = \mathrm{Lie}(T^n) \simeq \mathbb{R}^n$ its Lie algebra and by $\mathfrak{t}^*$ its dual. A *moment map*

$$\mu : X \to \mathfrak{t}^*$$

is a T^n-equivariant map satisfying

$$\iota_{\xi_X}\omega = \mathrm{d}\langle \mu, \xi \rangle \quad \text{for all } \xi \in \mathfrak{t},$$

where ξ_X is the vector field generated by ξ.

Definition 2.1 (Symplectic toric manifold)**.** *A* symplectic toric manifold *is a quadruple (X, ω, T^n, μ) consisting of a compact connected symplectic manifold (X, ω), an effective Hamiltonian action of T^n and an associated moment map $\mu : X \to \mathfrak{t}^*$.*

The fundamental convexity theorem of Atiyah and of Guillemin-Sternberg [At, GS] asserts that the image of the moment map is a convex polytope, and more precisely that

$$P := \mu(X) \subset \mathfrak{t}^* \simeq \mathbb{R}^n$$

is the convex hull of the images of the fixed points of the action. Moreover, the fibres of μ are connected and P is a compact convex polytope.

When (X, ω) is smooth and the action is effective, the polytope P satisfies extra integrality and smoothness conditions that can be read off from the local normal form of the Hamiltonian action near fixed points. To state these conditions, it is convenient to express P as an intersection of half-spaces.

A convex polytope $P \subset \mathbb{R}^n$ is called *simple* if exactly n facets meet at each vertex, and *rational* if its facets can be written as

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) \geq 0\}, \qquad \ell_k(x) = \langle \nu_k, x \rangle + \lambda_k,$$

for some $\nu_k \in \mathbb{Z}^n$ primitive inward-pointing normals and $\lambda_k \in \mathbb{R}$. The facets $F_k = \{\ell_k = 0\} \cap P$ correspond bijectively to the irreducible T^n-invariant divisors in any toric compactification of X; the primitive normal ν_k is the weight of the circle subgroup that fixes F_k pointwise.

Definition 2.2 (Delzant polytope)**.** *A polytope $P \subset \mathbb{R}^n$ is a* Delzant polytope *if:*

(1) *P is simple;*
(2) *P is rational, i.e.*

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) \geq 0\}, \qquad \ell_k(x) = \langle \nu_k, x \rangle + \lambda_k, \quad \nu_k \in \mathbb{Z}^n \textit{ primitive};$$

(3) *P is smooth: for each vertex $p \in P$ the inward normals $\nu_{k_1}, \ldots, \nu_{k_n}$ of the n facets meeting at p form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Delzant's theorem shows that the symplectic and combinatorial data are equivalent.

Theorem 2.3 (Delzant)**.** *The assignment $(X, \omega, T^n, \mu) \mapsto \mu(X)$ induces a bijection between:*

- *isomorphism classes of compact connected symplectic toric manifolds up to equivariant symplectomorphism;*
- *Delzant polytopes in $\mathbb{R}^n$ up to the action of the integral affine group $\mathrm{Aff}(\mathbb{Z}^n)$ (i.e. up to translations and linear transformations in $GL(n, \mathbb{Z})$).*

Thus, once a lattice in $\mathfrak{t}$ has been fixed, the polytope P encodes the symplectic toric manifold completely. Conversely, starting from a Delzant polytope P, one can reconstruct (X, ω, T^n, μ) by a symplectic reduction construction [De, Gu]: one considers the standard Hamiltonian action of T^N on $(\mathbb{C}^N, \omega_0)$, chooses an appropriate subtorus $K \subset T^N$ determined by the normals ν_k, and defines

$$X := \mu_K^{-1}(\eta)/K,$$

for a suitable level η of the K-moment map, equipped with the induced T^n-action and symplectic form. The moment map for the residual T^n-action has image P.

This classification provides the primary source of examples in the subject: projective toric varieties (such as $\mathbb{C}P^n$, Hirzebruch surfaces, and products thereof) correspond to Delzant polytopes with integer vertices.

2.2. **Complex structures and Kähler potentials.** A symplectic toric manifold (X, ω, T^n, μ) admits many compatible complex structures. A choice of an invariant integrable complex structure J compatible with ω turns (X, ω) into a toric Kähler manifold (X, ω, J) [Gu, Ab]. On the dense open orbit

$$X_0 \simeq (\mathbb{C}^*)^n,$$

we may use logarithmic coordinates

$$z_j = e^{y_j + i\theta_j}, \qquad y_j \in \mathbb{R},\ \theta_j \in \mathbb{R}/2\pi\mathbb{Z}, \quad j = 1, \ldots, n.$$

The action of T^n is by translation in the angular variables $\theta = (\theta_1, \ldots, \theta_n)$ and the Kähler metric is determined by a strictly convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, called a *(complex) Kähler potential*, via

$$\omega = i\partial\bar{\partial}\phi(y), \qquad g = \sum_{i,j} \phi_{ij}(y)\ \mathrm{d}y_i\, \mathrm{d}y_j + \sum_{i,j} \phi_{ij}(y)\ \mathrm{d}\theta_i\, \mathrm{d}\theta_j,$$

where $\phi_{ij} = \partial^2\phi/\partial y_i \partial y_j$. The associated moment map is

$$\mu = \nabla\phi : \mathbb{R}^n \to \mathbb{R}^n,$$

and its image is the interior P° of the moment polytope [Ab, Do1].

Different choices of ϕ give rise to Kähler metrics in the same cohomology class if and only if they differ by the real part of a holomorphic function on $(\mathbb{C}^*)^n$; at the level of T^n-invariant potentials this means that

$$\phi_2(y) = \phi_1(y) + \text{affine linear function in } y.$$

Thus, the space of T^n-invariant Kähler metrics in a fixed Kähler class can be identified with the space of strictly convex functions $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying suitable growth conditions at infinity modulo affine linear terms.

The *Legendre transform* provides a bridge from the complex picture on $\mathbb{R}^n_y$ to the symplectic picture on the polytope $P \subset \mathbb{R}^n_x$. Set

$$x = \nabla\phi(y) \in P^\circ, \qquad u(x) = \langle x, y \rangle - \phi(y),$$

so that $y = \nabla u(x)$ and

$$D^2 u(x) = (D^2\phi(y))^{-1}.$$

The function $u : P^\circ \to \mathbb{R}$ is strictly convex and is called the *symplectic potential* associated to the Kähler metric. It encodes the same metric in the following way: in coordinates $(x, \theta) \in P^\circ \times T^n$ one has

$$g_u = \sum_{i,j} u_{ij}(x)\ \mathrm{d}x_i\,\mathrm{d}x_j + \sum_{i,j} u^{ij}(x)\ \mathrm{d}\theta_i\,\mathrm{d}\theta_j, \qquad \omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\theta_i, \tag{2.1}$$

where $(u^{ij}) = (u_{ij})^{-1}$ [Ab, Do1].

Strict convexity of ϕ on $\mathbb{R}^n$ is equivalent to strict convexity of u on P°, and adding an affine linear function to ϕ corresponds to adding a constant to u. Thus, up to additive constants, the symplectic potential u parametrizes T^n-invariant Kähler metrics in a fixed Kähler class. In particular, the Abreu equation, which prescribes the scalar curvature of the metric, becomes a fourth-order PDE for u on the bounded polytope P, with natural boundary behaviour along ∂P.

2.3. **The Guillemin boundary condition.** The behaviour of the symplectic potential u near the boundary of P is dictated by the geometry of the torus action near the T^n-invariant divisors. Guillemin showed that the singular part of u is universal and depends only on the combinatorics of the polytope [Gu].

Write the Delzant polytope as

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\}, \qquad \ell_k(x) = \langle \nu_k, x\rangle + \lambda_k,$$

where each facet $F_k = \{\ell_k = 0\} \cap \overline{P}$ corresponds to a toric divisor $D_k \subset X$. Consider the model symplectic potential

$$u_0(x) := \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x). \tag{2.2}$$

Near a point of a facet F_k the defining function ℓ_k plays the role of a (signed) distance to the divisor, and the term $\ell_k \log \ell_k$ models the logarithmic singularity of the Kähler potential in complex coordinates; near a vertex, several ℓ_k's vanish simultaneously and u_0 encodes the joint singular behaviour.

Theorem 2.4 (Guillemin)**.** *Let (X, ω) be a compact symplectic toric manifold with moment polytope $P \subset \mathbb{R}^n$ written as above. Then any T^n-invariant Kähler metric on (X, ω) corresponds to a strictly convex function $u : P^\circ \to \mathbb{R}$ (the symplectic potential) such that*

$$u(x) - u_0(x) \in C^\infty(\overline{P}), \tag{2.3}$$

where u_0 is defined in (2.2)*. Conversely, any strictly convex function u on P satisfying* (2.3) *and a positivity condition on its Hessian defines a smooth T^n-invariant Kähler metric on X.*

The proof uses Delzant's symplectic reduction description and the local normal form of Hamiltonian torus actions. In a neighbourhood of a point of a divisor D_k, one can choose complex coordinates $(w_1, \dots, w_n)$ in which $D_k = \{w_1 = 0\}$ and the moment map coordinate x_1 is essentially $|w_1|^2/2$; in these coordinates a T^n-invariant Kähler potential has a local expansion

$$\phi_{\text{loc}} \sim x_1 \log x_1 + \text{smooth},$$

which translates via the Legendre transform into the $\ell_k \log \ell_k$ contribution in u_0. At a vertex where n divisors meet, one obtains a sum of such terms in independent directions. The fact that the remaining part $u - u_0$ extends smoothly to $\overline{P}$ reflects the smoothness of the metric across the toric divisors [Gu, Ab].

The function u_0 depends only on the combinatorial data $\{(\nu_k, \lambda_k)\}$ of the polytope and is therefore canonical. The difference

$$f(x) := u(x) - u_0(x)$$

is a smooth function on $\overline{P}$ and carries all the "free" degrees of freedom of the metric: prescribing $f \in C^\infty(\overline{P})$ satisfying a convexity condition yields a toric Kähler metric, and two metrics differ by a potential in $C^\infty(\overline{P})$.

Remark 2.5. *The same function u_0 appears naturally in the definition of the boundary measure $d\sigma$ on ∂P used in Donaldson's variational formulation of the Abreu equation: on a facet F_k defined by $\ell_k = 0$, $d\sigma$ is the $(n-1)$-dimensional Lebesgue measure induced by the lattice $\mathbb{Z}^n \cap \nu_k^\perp$, and integrals of u against $d\sigma$ in Donaldson's functional can be viewed as boundary contributions coming from the singular part u_0* [Do1].

Example 2.6 (Fubini-Study metric on $\mathbb{C}P^n$). *For $\mathbb{C}P^n$ with its standard toric structure, the moment polytope is the simplex*

$$\Delta_n = \Big\{ x \in \mathbb{R}^n : x_i \geq 0,\ \sum_{i=1}^n x_i \leq 1 \Big\}.$$

It can be written as

$$\Delta_n = \bigcap_{k=1}^{n+1} \{\ell_k(x) \geq 0\},$$

where $\ell_i(x) = x_i$ for $i = 1, \dots, n$ and $\ell_{n+1}(x) = 1 - \sum_{i=1}^n x_i$. The Fubini-Study metric corresponds to the symplectic potential

$$u_{\text{FS}}(x) = \sum_{i=1}^n x_i \log x_i + \Big(1 - \sum_{i=1}^n x_i\Big) \log\Big(1 - \sum_{i=1}^n x_i\Big),$$

which is exactly u_0 in this case, so that $f \equiv 0$ in (2.3).

3. Abreu's equation and extremal metrics

In the toric setting, the scalar curvature of a T^n-invariant Kähler metric can be written in a strikingly simple form in terms of the symplectic potential u on the moment polytope P. This leads to a fourth-order, fully nonlinear PDE—the Abreu equation—whose solutions correspond to metrics with prescribed scalar curvature, and in particular to extremal and constant scalar curvature Kähler metrics [Ab, Do1]. In this section we review Abreu's scalar curvature formula, the resulting PDE, and the characterization of extremal metrics in momentum coordinates.

3.1. **Abreu's scalar curvature formula.** Let u be a symplectic potential on P satisfying the Guillemin boundary condition (2.3). On the dense orbit $X_0 \simeq P^\circ \times T^n$ with coordinates (x, θ), the associated Kähler metric g_u and symplectic form ω take the block form

$$g_u = \sum_{i,j} u_{ij}(x)\ \mathrm{d}x_i\,\mathrm{d}x_j + \sum_{i,j} u^{ij}(x)\ \mathrm{d}\theta_i\,\mathrm{d}\theta_j, \qquad \omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\theta_i,$$

where $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ and $(u^{ij}) = (u_{ij})^{-1}$ [Ab, Do1]. In these coordinates the Riemannian volume form is

$$\mathrm{dVol}_{g_u} = \det(D^2 u)\ \mathrm{d}x\,\mathrm{d}\theta,$$

and the Levi-Civita connection and curvature can be expressed in terms of derivatives of u_{ij} and u^{ij}.

Abreu showed that, despite the apparent complexity of these expressions, the scalar curvature collapses to a remarkably simple divergence form.

Theorem 3.1 (Abreu)**.** *Let (X, ω, J) be a toric Kähler manifold with symplectic potential u on its moment polytope P. Then the scalar curvature S of the associated Kähler metric g_u can be written in momentum coordinates as*

$$S(u)(x) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}(x). \tag{3.1}$$

There are several complementary ways to derive (3.1).

(i) Direct computation in symplectic coordinates. One starts from the expression of g_u in (x, θ), computes the Christoffel symbols of g_u, and then the curvature tensor and scalar curvature. A key observation is that the nontrivial part of the connection is encoded in the derivatives $\partial_k u_{ij}$, and that these can be re-expressed in terms of derivatives of u^{ij} using the identity

$$\partial_k u^{ij} = -u^{ip}(\partial_k u_{pq}) u^{qj}.$$

After a lengthy but straightforward calculation, cancellations occur which leave only the double divergence of u^{ij} [Ab].

(ii) Legendre transform and complex coordinates. Alternatively, one can start from the complex Kähler potential ϕ in the y-coordinates and use the standard expression of the scalar curvature in terms of ϕ,

$$S(\phi) = -g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log\det\big(\phi_{\gamma\bar{\delta}}\big),$$

where $z_\alpha = y_\alpha + i\theta_\alpha$ and $\phi_{\gamma\bar{\delta}} = \partial^2\phi/\partial z_\gamma \partial \bar{z}_\delta$. Passing to the Legendre dual variable $x = \nabla\phi(y)$ and using the relation $D^2 u = (D^2\phi)^{-1}$, one can rewrite $S(\phi)$ as the double divergence of u^{ij}, arriving at (3.1) [Ab, Do1].

(iii) Divergence structure and invariance. A useful alternative form of (3.1) is

$$S(u) = -{u^{ij}}_{ij} = -\partial_i \partial_j u^{ij},$$

which makes it clear that the Abreu operator

$$\mathcal{A}(u) := -\partial_i \partial_j u^{ij}$$

is a fourth-order, quasilinear, elliptic operator in divergence form. It is invariant under adding affine linear functions to u and under affine transformations of P induced by elements of $\mathrm{Aff}(\mathbb{Z}^n)$. These invariances reflect the geometric fact that adding an affine term to u does not change the metric and that changing the integral affine structure corresponds to changing the lattice in the torus.

Remark 3.2. *The divergence structure of* (3.1) *is crucial both for variational formulations (integration by parts against test functions) and for weak solution theories: it allows one to interpret $S(u)$ as a distribution whenever u^{ij} has two weak derivatives. Moreover, in dimension* 2*, the operator $\mathcal{A}(u)$ admits a "conjugate" formulation in terms of the linearized Monge-Ampère operator, which plays an important role in Donaldson's interior estimates* [Do2]*.*

3.2. **The Abreu equation and its variational structure.** Given a prescribed scalar curvature function $K(x)$ on P, the *Abreu equation* is the fourth-order PDE

$$-\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}(x) = K(x) \quad \text{in } P^\circ, \tag{3.2}$$

for a convex function u on P, together with the Guillemin boundary condition (2.3). In most geometric applications,

- $K(x)$ is a constant (cscK metric);
- $K(x)$ is an affine function (Calabi extremal metric);
- more generally, K is a smooth function reflecting a prescribed curvature profile or arising as the Hamiltonian of a holomorphic vector field.

From the PDE viewpoint, (3.2) is a fully nonlinear, uniformly elliptic equation on P° as long as u is uniformly convex: there exist constants $0 < \lambda \le \Lambda$ such that

$$\lambda\,\mathrm{Id} \le D^2 u(x) \le \Lambda\,\mathrm{Id} \qquad \text{for all } x \in P^\circ.$$

The operator is fourth-order because u^{ij} itself contains second derivatives of u, and the equation involves two further derivatives. Nonetheless, it enjoys a close relation to the second-order Monge-Ampère equation via the identity

$$u^{ij} = \frac{1}{\det(D^2 u)} U^{ij},$$

where U^{ij} is the cofactor matrix of $D^2 u$. Writing $\phi = \det(D^2 u)$ and $U^{ij} = \phi\, u^{ij}$, one can recast (3.2) as the coupled second-order system

$$\begin{cases} \det D^2 u = \phi, \\ -U^{ij}\,\partial_{ij}(\phi^{-1}) = K(x), \end{cases}$$

where the second equation is a linearized Monge-Ampère equation for ϕ^{-1} with coefficients U^{ij} [Do2, CHLS]. This decomposition is often convenient in regularity theory (see Section 5), but for the variational theory it is more natural to keep the fourth-order formulation.

Variational formulation. The Abreu equation arises as the Euler-Lagrange equation of a natural functional on the space of symplectic potentials. Fix a smooth function $A \in C^\infty(\overline{P})$, and define the linear functional

$$L_A(u) = \int_{\partial P} u \; \mathrm{d}\sigma - \int_P A(x)\, u(x) \; \mathrm{d}x, \tag{3.3}$$

where $\mathrm{d}\sigma$ is the canonical boundary measure determined by the facet normals [Do1]. One also defines the nonlinear functional

$$F_A(u) = -\int_P \log\det D^2 u \; \mathrm{d}x + L_A(u), \tag{3.4}$$

on the space of symplectic potentials satisfying the Guillemin boundary condition.

The first variation of the determinant term is given by the classical identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log\det D^2(u + tv) = u^{ij} v_{ij},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(-\int_P \log\det D^2(u + tv) \; \mathrm{d}x \right) = -\int_P u^{ij} v_{ij} \; \mathrm{d}x.$$

Integrating by parts twice and using the Guillemin boundary behaviour to control boundary terms, one obtains

$$-\int_P u^{ij} v_{ij} \; \mathrm{d}x = \int_P \big(\partial_i \partial_j u^{ij}\big) v \; \mathrm{d}x - \int_{\partial P} (\cdots)\, v \; \mathrm{d}\sigma,$$

where the boundary term cancels precisely with the variation of $\int_{\partial P} u \; \mathrm{d}\sigma$ in $L_A(u)$ when one restricts to variations within the Guillemin class [Do1]. Thus the first variation of F_A in the direction v is

$$\delta F_A(u)(v) = \int_P \big(\partial_i \partial_j u^{ij} + A(x)\big) v \; \mathrm{d}x,$$

and the Euler-Lagrange equation $\delta F_A(u) = 0$ for all v is exactly

$$-\partial_i \partial_j u^{ij} = A(x),$$

that is, the Abreu equation (3.2) with $K = A$.

Proposition 3.3 (Abreu-Donaldson). *Critical points of F_A (under variations preserving the Guillemin boundary condition) satisfy the Abreu equation*

$$S(u) = A(x)$$

on P.

When A is chosen to be the average scalar curvature of a given Kähler class, F_A coincides (up to an additive constant) with the Mabuchi K-energy restricted to T^n-invariant metrics [Do1, Ma]. The linear functional L_A is then the differential of the K-energy along geodesic rays, and its positivity properties encode K-stability in the toric category; see Section 4.

Weak formulations. Because of its divergence structure, the Abreu operator admits a natural weak formulation: for u convex with $u^{ij} \in W^{1,1}_{\mathrm{loc}}(P^\circ)$, one can define $S(u)$ as a distribution by

$$\langle S(u), \varphi \rangle := \int_P u^{ij}\, \partial_{ij}\varphi \, \mathrm{d}x$$

for $\varphi \in C_c^\infty(P^\circ)$. The equation $S(u) = K$ then means

$$\int_P u^{ij}\, \partial_{ij}\varphi \, \mathrm{d}x = -\int_P K(x)\varphi(x)\, \mathrm{d}x, \qquad \forall \varphi \in C_c^\infty(P^\circ).$$

This is particularly useful when working with minimizers of F_A that are a priori only convex and not smooth, as in Donaldson's variational theory and in the PDE approach of Chen-Li-Sheng [Do1, CLS3].

3.3. **Extremal metrics in toric coordinates.** Calabi introduced extremal Kähler metrics as critical points of the *Calabi functional*

$$\mathcal{C}(g) = \int_X S(g)^2 \, \mathrm{dVol}_g$$

on the space of Kähler metrics in a fixed cohomology class [Ca]. The Euler-Lagrange equation for this functional is

$$\bar{\partial}\mathrm{grad}^{1,0} S = 0,$$

i.e. the $(1,0)$-gradient of the scalar curvature is a holomorphic vector field. Metrics satisfying this condition are called *extremal metrics*; constant scalar curvature metrics are the special case where the extremal vector field vanishes. In the general picture developed by Fujiki and Donaldson, extremal metrics arise as zeros of a moment map for the action of the identity component of the automorphism group on the space of Kähler potentials [Ma, Do1].

In the toric setting, this condition simplifies drastically. The complexified torus $(\mathbb{C}^*)^n$ acts holomorphically, and its Lie algebra corresponds to real Hamiltonian functions on X that are invariant under T^n; in momentum coordinates $x \in P$ such Hamiltonians are precisely affine functions $x \mapsto \langle \xi, x \rangle + c$. Abreu observed that the scalar curvature of a toric Kähler metric is extremal if and only if it is affine on P [Ab].

Theorem 3.4 (Abreu)**.** *Let (X, ω, J) be a toric Kähler manifold with symplectic potential u and scalar curvature $S(u)$ given by* (3.1)*. Then the metric is extremal in the sense of Calabi if and only if $S(u)$ is an affine function on P.*

Idea of the proof. If the metric is extremal, the gradient vector field $\mathrm{grad}^{1,0} S$ is holomorphic and T^n-invariant, so it lies in the complexified Lie algebra of T^n. Thus S coincides, up to an additive constant, with the Hamiltonian of some real holomorphic vector field generated by the torus, which in momentum coordinates is an affine linear function of x. Conversely, if $S(u)$ is affine in x, then its Hamiltonian vector field is a real holomorphic vector field in the torus Lie algebra, and one checks that $\mathrm{grad}^{1,0} S$ is holomorphic, so the metric is extremal [Ab, Do1]. □

Thus, in the toric case, extremal metrics correspond exactly to symplectic potentials u solving the Abreu equation (3.2) with right-hand side

$$K(x) = \langle \xi, x \rangle + c$$

for some $\xi \in \mathbb{R}^n$ and $c \in \mathbb{R}$, subject to the Guillemin boundary condition. The constant scalar curvature case corresponds to $\xi = 0$.

Example 3.5 (Toric Fano manifolds and Kähler-Einstein metrics)**.** *On a toric Fano manifold X with moment polytope P, a Kähler-Einstein metric corresponds to a solution of the Abreu equation with $K = \lambda$ constant and $[\omega] = c_1(X)$, together with an additional global volume constraint coming from the Einstein condition. In this case, the Futaki invariant can be expressed in terms of the barycenter of P, and Kähler-Einstein metrics exist precisely when the barycenter is at the origin in suitable coordinates* [WZh, Do1]*. In particular, for $\mathbb{C}P^n$ the standard simplex has barycenter at* $(1/(n+1), \ldots, 1/(n+1))$ *relative to the origin, and after an integral affine change of coordinates this corresponds to vanishing Futaki invariant and the existence of the Fubini-Study metric.*

Remark 3.6. *In non-toric settings, explicit extremal metrics are typically constructed using additional geometric structure (such as Hamiltonian* 2*-forms, fibration structures, or cohomogeneity-one symmetry)* [ACG]*. The toric case is special in that the extremal condition reduces to a global fourth-order PDE on a polytope with explicit boundary behaviour: this concreteness makes toric manifolds an ideal laboratory for testing conjectures about extremal metrics, stability, and energy functionals, and underlies much of Donaldson's program discussed in Section 4.*

4. Donaldson's program on toric surfaces

In a remarkable series of papers [Do1, Do2, Do3, Do4], Donaldson developed a program for solving the constant scalar curvature (and more generally extremal) Kähler metric problem on toric surfaces by translating it into a fourth-order real Monge-Ampère-type equation on the moment polytope and relating existence of solutions to a toric notion of K-stability. This provides a complete bridge, in complex dimension two, between algebro-geometric stability and the analysis of the Abreu equation with Guillemin boundary conditions, and can be viewed as an early verification of the Yau-Tian-Donaldson picture in a highly nontrivial class of examples [Do1, Do3, Do4].

Let (X, L) be a polarized toric surface, and let $P \subset \mathbb{R}^2$ be the corresponding Delzant polygon with boundary measure σ and Lebesgue measure μ. A T-invariant Kähler metric in $c_1(L)$ corresponds to a symplectic potential u on P in the Guillemin class, and its scalar curvature is given by Abreu's formula

$$S(u) = -\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \, \partial x_j},$$

where (u^{ij}) is the inverse Hessian of u [Ab]. The extremal (in particular constant scalar curvature) condition is then the fourth-order PDE

$$S(u) = A,$$

for a prescribed smooth function A on P (constant in the cscK case), together with the Guillemin boundary conditions along ∂P [Gu, Do1].

Donaldson's program can be organized in four main steps, each corresponding to one of the papers above:

- identify the Mabuchi functional on toric varieties with a real functional F_A on convex functions on P and relate its coercivity to a toric K-stability condition [Do1];
- establish interior a priori estimates for solutions of Abreu's equation on general convex domains [Do2];
- obtain global a priori control under a geometric "M-condition" and set up a continuity method for extremal solutions on toric surfaces [Do3];
- combine stability and a priori estimates to prove an existence theorem for constant scalar curvature metrics on toric surfaces [Do4].

We now describe the main results and ideas of each paper in turn.

4.1. **Scalar curvature and stability of toric varieties.** The paper [Do1] introduces a notion of K-stability for a general polarized variety (V, L), formulated in terms of test configurations and Futaki invariants, and proposes the Yau-Tian-Donaldson conjecture that K-stability is equivalent to the existence of a constant scalar curvature Kähler metric in $c_1(L)$. In the toric setting, this abstract framework becomes very concrete and can be expressed entirely in terms of convex geometry on the moment polytope.

Let $P \subset \mathbb{R}^2$ be the Delzant polygon associated to (X, L). Donaldson considers the space $\mathcal{S}_P$ of symplectic potentials u on P satisfying the Guillemin boundary condition and normalized, for instance, by fixing $u(p_0) = 0$ at a point $p_0 \in P^\circ$ and $\int_{\partial P} u \, \mathrm{d}\sigma = 1$ [Do1]. For any bounded measurable function A on P, he defines the linear functional

$$L_A(u) = \int_{\partial P} u \, \mathrm{d}\sigma - \int_P A\, u \, \mathrm{d}\mu$$

and the nonlinear functional

$$F_A(u) = -\int_P \log\det(D^2 u) \, \mathrm{d}\mu = L_A(u),$$

which is the Mabuchi K-energy restricted to T-invariant metrics when A is chosen to be the average scalar curvature [Ma, Do1]. The Euler-Lagrange equation of F_A is precisely Abreu's equation $S(u) = A$ on P.

On the algebro-geometric side, Donaldson studies test configurations of (X, L) which are themselves toric. Such test configurations are in one-to-one correspondence with rational convex piecewise linear functions f on P with integral slopes: to each f one associates a polytope in $\mathbb{R}^3$ obtained by taking the region under the graph of f over P, whose lattice geometry encodes the total weight of a $\mathbb{C}^*$-action on the spaces $H^0(X, L^k)$ [Do1]. Applying equivariant Riemann-Roch, he computes the asymptotics

$$\dim H^0(X, L^k) = a_0 k^2 + a_1 k + a_2 + O(k^{-1}), \quad w_k(f) = b_0 k^3 + b_1 k^2 + b_2 k + O(1),$$

where $w_k(f)$ is the total weight of the induced $\mathbb{C}^*$-action, and shows the Donaldson-Futaki invariant of the test configuration defined by f is proportional to $L_A(f)$ [Do1]. Thus, the algebro-geometric notion of K-stability for toric test configurations is translated into the convex-analytic condition

$$L_A(f) \geq 0 \quad \text{for all rational convex piecewise linear } f,$$

with equality only when f is affine linear (corresponding to product test configurations).

From the analytic viewpoint, F_A is convex along line segments in $\mathcal{S}_P$ and behaves linearly along rays associated to piecewise linear perturbations. Donaldson proves that if $L_A(f) \geq 0$ for all rational convex piecewise linear f and $L_A(f) = 0$ only for affine linear functions, then:

- F_A is bounded below on $\mathcal{S}_P$; and
- any minimizing sequence u_ℓ for F_A admits a subsequence which converges, after subtracting suitable affine linear functions, to a convex generalized minimizer of F_A [Do1].

The limit u_∞ is in general only convex and may fail to be smooth; it solves Abreu's equation in a weak sense. This gives one direction of the YTD picture for toric surfaces: toric K-stability (with respect to toric degenerations) implies the existence of a generalized minimizing metric for the Mabuchi functional [Do1].

An important by-product of the analysis is the construction of explicit destabilizing test configurations: by carefully choosing piecewise linear functions f for which $L_A(f) < 0$, Donaldson produces examples of polarized toric surfaces which are K-unstable and therefore admit no constant scalar curvature metric in the given polarization [Do1]. This illustrates concretely how convex geometry of P detects the obstruction to canonical metrics.

4.2. **Interior estimates for solutions of Abreu's equation.** The second paper [Do2] is devoted to purely analytic questions for Abreu's equation on general convex domains and plays a key role in the later compactness and continuity arguments. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain (in particular a polytope), let σ be a positive measure on $\partial\Omega$, and consider convex functions u solving

$$S(u) = -\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i\, \partial x_j} = A(x)$$

with prescribed boundary behaviour of Guillemin type [Do2]. The central goal is to obtain interior a priori estimates for u and its derivatives under geometric hypotheses on the triple (Ω, σ, A), reflecting a uniform form of K-stability.

Donaldson works in a function space $\mathcal{S}_{\Omega,\sigma}$ of convex functions on Ω with controlled asymptotics near the boundary (modeled on the Guillemin singularity), and defines a normalized class $\tilde{\mathcal{C}}$ of convex functions v on Ω satisfying $v \geq 0$, $v(p_0) = 0$ at a fixed interior point p_0, and $\int_{\partial\Omega} v \,\mathrm{d}\sigma = 1$ [Do2]. For such v he considers the linear functional

$$L_A(v) = \int_{\partial\Omega} v \,\mathrm{d}\sigma - \int_{\Omega} A(x)\, v(x)\, \mathrm{d}x,$$

in exact analogy with the toric situation, and introduces the following quantitative stability condition.

> *Condition 1* (stability of (Ω, σ, A)). There exists a constant $\kappa > 0$ such that
>
> $$L_A(v) \geq \kappa \int_{\partial\Omega} v \,\mathrm{d}\sigma$$
>
> for all $v \in \tilde{\mathcal{C}}$.

This is a uniform version of $L_A \geq 0$ on normalized convex functions and can be regarded as an analytic form of uniform K-stability for the data (Ω, σ, A) [Do2]. Donaldson proves that Condition 1 is necessary for the existence of a smooth solution of Abreu's equation in $\mathcal{S}_{\Omega,\sigma}$, and more importantly, that it yields strong interior estimates.

The main result of [Do2] in real dimension two can be stated schematically as follows.

Theorem 4.1 (Donaldson [Do2])**.** *Suppose $n = 2$ and let A be a smooth bounded function on Ω. If (Ω, σ, A) satisfies Condition 1, then any normalized solution $u \in \mathcal{S}_{\Omega,\sigma}$ of Abreu's equation $S(u) = A$ satisfies uniform interior estimates of all orders.*

More precisely, for each compact $\Omega' \Subset \Omega$ *and each* $k \geq 0$ *there exists a constant* C_k*, depending only on* (Ω, σ, A) *and* $\operatorname{dist}(\Omega', \partial\Omega)$*, such that*

$$\|u\|_{C^k(\Omega')} \leq C_k.$$

In particular, the Riemannian metric g_u *associated to* u *has curvature and all its derivatives uniformly bounded on* Ω'*.*

The proof combines several techniques from the theory of fully nonlinear elliptic equations:

- *Determinant bounds.* Using the variational structure of F_A and maximum principle arguments applied to suitable auxiliary quantities, Donaldson first derives lower and upper bounds on $\det(D^2u)$ in terms of the distance to the boundary [Do2]. In dimension two he obtains estimates of the form

  $$C^{-1}\, d(x, \partial\Omega)^{-2} \leq \det D^2u(x) \leq C\, d(x, \partial\Omega)^{-2}$$

  for x away from $\partial\Omega$.
- *Linearized Monge-Ampère theory.* Setting U^{ij} to be the cofactor matrix of D^2u, one has $u^{ij} = U^{ij}/\det D^2u$ and Abreu's equation can be written as

  $$-\partial_i\partial_j u^{ij} = -\partial_i\partial_j(\phi^{-1}U^{ij}) = A(x),$$

  where $\phi = \det D^2u$ [Ab, Do2]. The linearized operator

  $$\mathcal{L}v := U^{ij}\partial_i\partial_j v$$

  is a second-order elliptic operator with measurable coefficients. Using the determinant bounds and the structure of the "sections" of convex functions, Donaldson applies the regularity theory for the linearized Monge-Ampère equation to obtain Hölder and then higher regularity for u [Do2].
- *Two-dimensional structure.* In real dimension two, the operator $\mathcal{L}$ admits a "conjugate" harmonic function and a potential formulation reminiscent of complex analysis. Donaldson exploits this to construct special coordinates adapted to u and to sharpen the interior estimates, obtaining gradient bounds for the curvature tensor and controlling the metric in a scale-invariant way [Do2].

These interior estimates are independent of any specific toric geometry: they apply to general convex domains and right-hand sides satisfying Condition 1. In the toric situation, they provide exactly the control needed on compact subsets of the moment polygon once one has uniform boundary and global geometric control from stability and the M-condition.

4.3. **Extremal metrics on toric surfaces: a continuity method.** The third paper [Do3] brings the analytic and algebro-geometric ingredients together in the toric surface case. Donaldson considers triples

$$(P, \sigma, A),$$

where P is a Delzant polygon, σ is the standard boundary measure, and A is a smooth function on P representing the desired scalar curvature (constant in the cscK case, affine linear for extremal metrics). A symplectic potential u in the Guillemin class solves the extremal (Abreu) equation

$$S(u) = A$$

if and only if the associated T-invariant Kähler metric on the toric surface has scalar curvature A [Ab, Do1].

A key new notion introduced in [Do3] is the *M-condition*, a quantitative geometric condition on the Riemannian metric

$$g_u = \sum_{i,j} u_{ij}\ \mathrm{d}x_i\ \mathrm{d}x_j + \sum_{i,j} u^{ij}\ \mathrm{d}\theta_i\ \mathrm{d}\theta_j$$

on $P \times \mathbb{T}^2$ [Do3]. For a convex function u on P° and any two distinct points $p, q \in P^\circ$, let ν be the unit vector pointing from p to q, and define

$$V(p,q) = (\nabla_\nu u)(q) - (\nabla_\nu u)(p),$$

the change of directional derivative of u along the segment $[p,q]$. One enlarges the segment $[p,q]$ symmetrically about its midpoint to a longer segment $I(p,q)$; roughly speaking, $I(p,q)$ is obtained by extending $[p,q]$ by a factor 3 so that its midpoint stays fixed [Do3].

Definition 4.2 (Donaldson's M-condition [Do3])**.** *A convex function u on P satisfies the M-condition if there exists a constant $M > 0$ such that*

$$V(p,q) \le M$$

whenever $p, q \in P^\circ$ are such that the extended segment $I(p,q)$ is contained in P.

Geometrically, the M-condition gives a uniform control on how rapidly the gradient of u can change along segments that stay away from the boundary. In terms of the associated Kähler metric g_u, it implies uniform bounds on the lengths of geodesic segments in P (with respect to the metric $\sum u_{ij}\ \mathrm{d}x_i\, \mathrm{d}x_j$) in terms of their Euclidean lengths and prevents the formation of narrow necks or cusps in the metric [Do3].

The main analytic results of [Do3] can be summarized as follows. Consider a sequence of data $(P^{(\alpha)}, \sigma^{(\alpha)}, A^{(\alpha)})$ converging in a natural sense to (P, σ, A), and a sequence of symplectic potentials $u^{(\alpha)}$ solving $S(u^{(\alpha)}) = A^{(\alpha)}$ on $P^{(\alpha)}$, normalized by $u^{(\alpha)}(p_0) = 0$ for some fixed interior point p_0. Assume that

(1) each $u^{(\alpha)}$ satisfies a uniform M-condition;
(2) the functions $A^{(\alpha)}$ are uniformly bounded in C^k on $P^{(\alpha)}$ for each k;
(3) the polygons $P^{(\alpha)}$ converge to P in the Hausdorff topology with controlled combinatorics (no facets disappearing in the limit).

Then, after passing to a subsequence, the functions $u^{(\alpha)}$ converge in C^∞ on compact subsets of P° to a symplectic potential u solving $S(u) = A$ [Do3].

The proof proceeds in two main stages:

- *Blow-up analysis.* Assuming that curvature or higher derivatives of $u^{(\alpha)}$ blow up somewhere, Donaldson rescales around points of large curvature to obtain a sequence of rescaled metrics with bounded curvature and nontrivial limit. The interior estimates from [Do2], together with the M-condition (which is preserved under rescaling), imply that any blow-up limit is a complete scalar-flat Kähler metric on $\mathbb{R}^4$ (or on a half-plane or strip in the momentum coordinates) with strong geometric control [Do2, Do3]. Using a detailed classification of such limits in the toric setting, Donaldson shows that no nonflat complete scalar-flat Kähler metric satisfying the appropriate asymptotics and M-condition can exist, leading to a contradiction. Hence curvature and all derivatives remain uniformly bounded on compact subsets.
- *Continuity method.* For a fixed combinatorial type of polygon P, one considers a path of scalar curvature functions

$$A_t = (1-t)A_0 + tA_1, \qquad t \in [0,1],$$

  where A_0 is a reference function for which a solution is known (for instance the scalar curvature of a standard toric metric) and A_1 is the desired extremal function [Do3]. The openness of the set of t for which a solution exists follows from the implicit function theorem applied to the Abreu operator, using the Fredholm properties of the linearized operator in the toric setting. Closedness along the path is obtained by verifying that the solutions satisfy a uniform M-condition and then applying the compactness result just described.

Even though [Do3] is formulated for general extremal metrics (non-constant A), its technical output is a robust compactness and a priori estimate theory for solutions of Abreu's equation on toric surfaces under the M-condition. This machinery is the main analytic ingredient in the later existence theorem for constant scalar curvature metrics on toric surfaces [Do4].

4.4. **Constant scalar curvature metrics on toric surfaces.** The final paper [Do4] completes the program in the constant scalar curvature case for toric surfaces. Working again with a Delzant polygon P and the associated toric surface (X, L), Donaldson considers the cscK equation

$$S(u) = \bar{S},$$

where $\bar{S}$ is the topologically determined average scalar curvature of the class $c_1(L)$, and u is a symplectic potential in the Guillemin class [Ab, Do1].

On the algebro-geometric side, he refines the notion of toric K-stability to a *relative* version (to account for the presence of holomorphic vector fields). In the toric setting, holomorphic vector fields commuting with the torus action correspond to affine linear functions ℓ on P, and the Futaki invariant of such a vector field is encoded by $L_{\bar{S}}(\ell)$ [Do1, Do4]. Relative K-stability requires that

$$L_{\bar{S}}(f) \geq 0$$

for all rational convex piecewise linear f on P, with equality only for those f differing from a fixed affine linear function by an additive constant (corresponding to the extremal vector field). This yields a purely convex-geometric stability condition on the polygon P [Do4].

On the analytic side, we know that Donaldson combines the variational formulation of the Mabuchi functional, the interior estimates from [Do2], and the global compactness theory under the M-condition from [Do3]. The strategy is to construct a continuity path

$$S(u_t) = (1-t)A_0 + t\bar{S}, \qquad t \in [0,1],$$

where A_0 is the scalar curvature of a reference toric metric. One shows that:

- the set of t for which there exists a solution u_t is nonempty (for small t) and open, by standard implicit function arguments; and
- if (X, L) is relatively K-stable in the toric sense, then along the path one has uniform stability estimates of the type in Condition 1 for the triples (P, σ, A_t) and uniform M-bounds for the corresponding solutions u_t [Do2, Do3, Do4].

The stability estimates imply, via [Do2], uniform interior bounds for u_t and its derivatives on compact subsets of P. The M-condition, together with the blow-up analysis of [Do3], prevents curvature concentration and ensures global compactness. If one assumes that the path cannot be continued past some $t^* < 1$, then blowing up a sequence u_{t_k} with $t_k \nearrow t^*$ produces a nontrivial complete scalar-flat Kähler limit solving the Abreu equation on a limiting domain, which in turn yields a destabilizing convex function on P, contradicting relative K-stability [Do3, Do4].

The main theorem of [Do4] can be stated schematically as follows.

Theorem 4.3 (Donaldson [Do4])**.** *Let (X, L) be a toric surface with Delzant polygon P and average scalar curvature $\bar{S}$. Assume that (X, L) is relatively K-stable with respect to toric degenerations, i.e. that $L_{\bar{S}}(f) \geq 0$ for all rational convex piecewise linear functions f on P, with equality only for those f corresponding to holomorphic vector fields. Then there exists a smooth symplectic potential u in the Guillemin class solving*

$$S(u) = \bar{S}$$

on P. In particular, X admits a T-invariant constant scalar curvature Kähler metric in the class $c_1(L)$.

In other words, for toric surfaces, relative toric K-stability is sufficient for the existence of a cscK metric, giving a toric version of the Yau-Tian-Donaldson conjecture in complex dimension two. Together with the general existence results of Chen-Donaldson-Sun for Kähler-Einstein metrics on Fano manifolds [CDS1, CDS2, CDS3] and the later work of Chen-Cheng on the cscK equation on arbitrary Kähler manifolds [CC1, CC2, CC3], Donaldson's program on toric surfaces provides a model case where algebro-geometric stability, variational methods, and fine PDE estimates for the Abreu equation can be seen working together in full detail.

To summarize, Donaldson's four toric papers build a complete bridge, in the surface case, from algebro-geometric stability (expressed as positivity properties of the functional L_A on convex functions on the moment polygon) to the existence of smooth solutions of Abreu's equation with Guillemin boundary conditions, and hence to the existence of extremal and constant scalar curvature Kähler metrics. The key analytic tools are the interior estimates for Abreu's equation under a quantitative stability condition, the global M-condition controlling the geometry of the metric near the boundary of the polytope, and a delicate blow-up analysis ruling out curvature concentration [Do2, Do3, Do4].

5. The Monge-Ampère equation with Guillemin boundary conditions

5.1. **From Abreu's equation to a second-order system.** The Abreu equation can be viewed as a fourth-order scalar equation in divergence form, but it is often advantageous to regard it as a coupled second-order system involving the real Monge-Ampère operator and its linearization. Let u be a symplectic potential on a simple polytope $P \subset \mathbb{R}^n$ satisfying the Guillemin boundary condition, and set

$$u_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad (u^{ij}) = (u_{ij})^{-1}, \qquad \phi(x) = \det D^2 u(x).$$

Let U^{ij} denote the cofactor matrix of $D^2 u$,

$$U^{ij} = \mathrm{cofactor}_{ij}(D^2 u) = \phi\, u^{ij}.$$

Then the scalar curvature can be written as

$$S(u) = -\partial_i \partial_j u^{ij} = -\frac{1}{\phi}\, \partial_i\big(U^{ij} \partial_j (\log \phi)\big) = -U^{ij} \partial_{ij}(\phi^{-1}),$$

which leads to the following system for the pair (u, ϕ):

$$\begin{cases} \det D^2 u = \phi, \\ -U^{ij}\, \partial_{ij}(\phi^{-1}) = K(x). \end{cases} \tag{5.1}$$

Here the first equation is a (real) Monge-Ampère equation prescribing the determinant of the Hessian of u, while the second is a linearized Monge-Ampère equation for ϕ^{-1} with coefficients U^{ij} [Do2, CHLS]. The ellipticity of the system is encoded in the uniform convexity of u, which ensures that $D^2 u$ and U^{ij} are positive definite.

Writing the Abreu equation in the form (5.1) has several conceptual and technical advantages. First, it separates the highly nonlinear determinant constraint from the (formally) linear curvature equation, making it natural to attack the problem in two stages: solve a Monge-Ampère equation with prescribed right-hand side, and then understand the linear equation satisfied by ϕ^{-1}. Second, it clarifies the role of the cofactor matrix U^{ij} as the metric tensor for the linearized operator, and thus connects the analysis of Abreu's equation to the rich regularity theory for the linearized Monge-Ampère equation. Finally, in toric geometry the structure of the right-hand side of the

Monge-Ampère equation is particularly rigid, which leads to the Guillemin boundary problem discussed below.

In the toric setting, a T^n-invariant Kähler metric in a fixed Kähler class corresponds to a symplectic potential u satisfying the Guillemin boundary condition on a simple polytope $P \subset \mathbb{R}^n$, and the Kähler class itself determines the asymptotics of the associated volume form in momentum coordinates. A straightforward computation starting from the model potential

$$u_0(x) = \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x), \qquad P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\},$$

shows that

$$\det D^2 u_0(x) \sim \frac{c}{\prod_{k=1}^{N} \ell_k(x)} \quad \text{as } x \to \partial P,$$

for a suitable positive constant c depending only on the combinatorics of P [Gu, Ab]. If we now write $u = u_0 + f$ with $f \in C^\infty(\overline{P})$, the determinant takes the form

$$\det D^2 u = \frac{h(x)}{\prod_{k=1}^{N} \ell_k(x)},$$

where $h \in C^\infty(\overline{P})$ is strictly positive. Thus the natural Monge-Ampère equation associated to a toric Kähler class is precisely of the form

$$\det D^2 u = \frac{h(x)}{\prod_{k=1}^{N} \ell_k(x)}, \tag{5.2}$$

coupled with the Guillemin boundary condition on u. Equation (5.2) is the starting point for the theory of the Monge-Ampère equation with Guillemin boundary conditions.

5.2. **The Guillemin boundary problem.** Motivated by the toric situation, one is led to isolate the following purely real boundary value problem.

Definition 5.1 (Guillemin boundary problem for Monge-Ampère equations)**.** *Let $P \subset \mathbb{R}^n$ be a simple polytope*

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\}, \qquad \ell_k(x) = \langle \nu_k, x \rangle + \lambda_k,$$

and let $h \in C^\infty(\overline{P})$ be strictly positive. The Guillemin boundary problem *is*

$$\begin{cases} \det D^2 u = \dfrac{h(x)}{\prod_{k=1}^{N} \ell_k(x)} & \text{in } P^\circ, \\ u(x) - \displaystyle\sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}). \end{cases} \tag{5.3}$$

The right-hand side in (5.3) blows up like the reciprocal of the product of the distances to the facets of P, so the Monge-Ampère equation is singular at the boundary: near a point on a facet $\{\ell_k = 0\}$ it looks like $\det D^2 u \sim \ell_k^{-1}$, and near a vertex where m facets meet it behaves like the product of m such singular factors. The boundary condition, on the other hand, prescribes that u has exactly the logarithmic singularity $\sum \ell_k \log \ell_k$ corresponding to the model potential u_0, and is otherwise smooth up to the boundary.

From the geometric viewpoint, (5.3) describes exactly those symplectic potentials that induce smooth toric Kähler metrics with prescribed volume form in a given Kähler class. From the analytic viewpoint, it provides a natural model for fully nonlinear equations with singular coefficients on domains with corners: the polytope P has a stratified boundary (facets, edges, vertices, and higher-codimension faces in higher dimensions), and the singularity of the equation degenerates along this stratification. Establishing existence, uniqueness, and regularity for (5.3) thus requires combining techniques from the theory of Monge-Ampère equations with a careful analysis of the polyhedral geometry of P [Ru, Hu, HS, BST]. In a forthcoming work with Zhou, an analogous Guillemin boundary value problem will be formulated and analyzed for the full Abreu equation, providing a fourth-order counterpart of (5.3) in the toric setting [WZ].

5.3. **Rubin's pioneering work in dimension two.** The first systematic PDE study of (5.3) is due to Rubin [Ru]. He considers the case $n = 2$ with P a bounded convex polygon and $h \in C^\infty(\overline{P})$ positive, and constructs convex solutions satisfying the Guillemin behaviour along edges. The main results can be summarized as follows:

- for any admissible boundary data (compatible with the Guillemin asymptotics along edges) there exists a convex Alexandrov solution u of (5.3);
- away from the vertices of P, i.e. on compact subsets of an open edge or in the interior of P, the solution is smooth and enjoys precise asymptotic expansions of the form
$$u(x, y) = y \log y + a(x) + b(x)\, y + O(y^{1+\beta}),$$
in coordinates (x, y) adapted to the edge, for some $\beta \in (0, 1)$ and smooth functions a, b [Ru].

In particular, the logarithmic singularity is shown to be universal along each edge, and the remaining terms have controlled Hölder regularity.

Rubin's analysis relies on several key ideas. First, he studies model equations of the form
$$\det D^2 u = y^{-1}$$
in a half-ball, which capture the local behaviour near a straight edge. Second, he uses partial Legendre transforms in the tangential direction to convert the Monge-Ampère equation into a quasilinear equation better adapted to boundary regularity. Third, he constructs explicit barrier functions and uses perturbative arguments to obtain global existence and regularity away from corners. The delicate behaviour at the vertices of

P is left as an open problem: the interaction of the singularities associated to several edges meeting at a point is substantially more complicated.

Rubin's work shows that, at least away from corners, the Guillemin boundary condition is analytically compatible with the singular right-hand side of (5.3): the solution u exists, is unique in an appropriate class, and is as regular as one can reasonably expect given the singular nature of the equation.

5.4. **Huang's solution of the polygonal case.** In a major breakthrough, Huang solved the Guillemin boundary problem completely in dimension two [Hu]. Let $P \subset \mathbb{R}^2$ be a convex polygon written as

$$P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\},$$

where each ℓ_k is affine and $\{\ell_k = 0\} \cap P$ is an edge of P. Let $h \in C^\infty(\overline{P})$ be strictly positive and consider the boundary value problem

$$\det D^2 u = h(x) \prod_{k=1}^{N} \ell_k(x)^{-1} \quad \text{in } P, \qquad u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}). \tag{5.4}$$

Rubin [Ru] showed that, under a natural family of compatibility conditions on h (expressed in terms of integrals of h along edges and around the vertices), there is a unique convex solution of (5.4) whose "smooth part" $u - \sum \ell_k \log \ell_k$ is C^∞ up to each open edge and Hölder continuous on $\overline{P}$, but he left the regularity at the vertices open. Huang proves that, under the same compatibility conditions, the solution is in fact smooth up to *all* vertices.

More precisely, given arbitrary vertex values $\{\alpha_i\}_{i=1}^N$, Huang shows that there exists a unique convex solution u of (5.4) with

$$u(p_i) = \alpha_i \quad (i = 1, \dots, N)$$

such that

$$u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}).$$

Thus in the polygonal case the universal singularity $\sum \ell_k \log \ell_k$ completely captures the boundary behaviour of u, and the remaining part extends smoothly across every corner.

The proof uses Rubin's global existence and edge regularity as a black box and then performs a detailed analysis near a fixed vertex. After an affine change of variables one may assume that a given vertex is at the origin $p_1 = 0$, and that the two edges meeting there are

$$\{x_1 = 0\} \cap P, \qquad \{x_2 = 0\} \cap P,$$

so that $\ell_1(x) = x_2$ and $\ell_N(x) = x_1$. In a small quarter-square

$$Q_1 = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\} \subset P$$

one writes

$$u(x) = x_1 \log x_1 + x_2 \log x_2 + v(x),$$

and absorbs the remaining factors $\ell_2, \dots, \ell_{N-1}$ of the denominator into a smooth, strictly positive function $h(x)$. In these local coordinates the equation becomes

$$(x_1 v_{11} + 1)(x_2 v_{22} + 1) - x_1 x_2 v_{12}^2 = h(x) \quad \text{in } Q_1, \tag{5.5}$$

with $h \in C^\infty(Q_1)$ and $h > 0$. The compatibility conditions show that if v is C^2 then necessarily

$$h(0) = 1, \qquad v_{11}(x_1, 0) = \frac{h(x_1, 0) - 1}{x_1}, \quad v_{22}(0, x_2) = \frac{h(0, x_2) - 1}{x_2},$$

and they determine uniquely the restrictions $v(x_1, 0)$ and $v(0, x_2)$ as smooth functions of one variable. Thus the regularity problem at the vertex reduces to proving that the solution v of the Dirichlet problem

$$\begin{cases} (x_1 v_{11} + 1)(x_2 v_{22} + 1) - x_1 x_2 v_{12}^2 = h(x) \quad \text{in } Q_1, \\ v(x_1, 0) = F(x_1), \quad v(0, x_2) = G(x_2) \end{cases}$$

is in $C^\infty(\overline{Q}_1)$, where F, G are the smooth boundary traces fixed by the data and, after subtracting an affine function, one has

$$v(0) = |Dv(0)| = 0.$$

Huang's argument proceeds in several stages.

- **Lipschitz and C^1 regularity at the corner.** Using Rubin's edge estimates, Huang first constructs explicit barrier functions that compare u with suitable quadratic perturbations of $x_1 \log x_1 + x_2 \log x_2 + v(x_1, 0)$ (respectively $+v(0, x_2)$) in small rectangles touching the sides. This yields two one-sided estimates of the form

  $$|v(x) - v(x_1, 0)| \le C x_2, \qquad |v(x) - v(0, x_2)| \le C x_1 \quad \text{in } Q_1,$$

  and shows that $v \in C^{0,1}(Q_1)$.

  To upgrade this to C^1, he considers the rescaling

  $$v_\lambda(x) = \frac{1}{\lambda} v(\lambda x), \qquad x \in \lambda^{-1} Q_1, \quad 0 < \lambda < 1.$$

  The Lipschitz bound implies $|v_\lambda(x)| \le C|x|$ on compact subsets of the quarter-plane $H = \{x_1 > 0, x_2 > 0\}$, and v_λ satisfies the same type of equation as (5.5) with right-hand side $h(\lambda x) \to h(0) = 1$. Passing to a subsequential limit gives a global solution V of the model corner equation

  $$(x_1 V_{11} + 1)(x_2 V_{22} + 1) - x_1 x_2 V_{12}^2 = 1 \quad \text{in } H, \qquad V|_{\{x_1=0\} \cup \{x_2=0\}} = 0,$$

  with growth $|V(x)| = o(|x| \log |x|)$ as $|x| \to \infty$. A carefully constructed logarithmic barrier then shows that the only such solution is $V \equiv 0$ (a Liouville-type theorem). The blow-up argument by contradiction using this theorem yields

$$v \in C^1(Q_1) \quad \text{and} \quad \lim_{x\to 0} \frac{|Dv(x)|}{|x|} = 0.$$

- **Quadratic growth and refined asymptotics.** With C^1 regularity at hand, Huang goes back to barrier constructions and shows that

$$|v(x)| \le C|x|^2 \quad \text{for } x \text{ near } 0,$$

together with boundary derivative estimates such as $|v_1(0, x_2)| \le Cx_2$ and $|v_2(x_1, 0)| \le Cx_1$.

Although this suggests the "natural" scaling $v(\lambda x)/\lambda^2$, the resulting equation is no longer of the same structural form as (5.5). Instead Huang keeps the linear scaling $v_\lambda(x) = v(\lambda x)/\lambda$ and uses a delicate iteration scheme to show that, on annuli $\{x : c \le |x| \le 2c\}$, all derivatives of v_λ are small, with estimates of the form

$$|D^k v_\lambda(x)| \le C_{k,a}\lambda^a \quad \text{for any } a \in (0, \tfrac{1}{2}),$$

uniformly in x away from the axes. Translated back to the original scale, this yields refined asymptotics

$$|D^k v(x)| \le C_{k,a}|x|^{a+1-k}, \qquad a \in (0, \tfrac{1}{2}),$$

which already imply that D^2v remains bounded and that x_1D^3v, x_2D^3v are Hölder continuous.

- **Control of the mixed derivative and $C^{2,\alpha}$-regularity.** A key further step is to study the mixed derivative $V = v_{12}$. Differentiating (5.5) twice leads to a second-order linear equation of the form

$$x_1V_{11} + V_1 + x_2V_{22} + V_2 = f(x) \quad \text{in } Q_1 \setminus \{0\},$$

where f is expressed in terms of h and the lower-order derivatives of v. To exploit this structure, Huang performs a sequence of explicit changes of variables which rewrite this equation as a Poisson equation

$$\Delta_y V(y) = f(y) \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^4,$$

with $f(y)$ decaying like a negative power of $|y|$. A barrier argument for the Laplacian in $\mathbb{R}^4$ then gives, for every $a > 0$,

$$|v_{12}(x)| \le C_a|x|^{-a}$$

near the corner. Integrating this information and combining it with the previously obtained estimates for v_{11} and v_{22}, he derives optimal pointwise bounds for Dv and higher derivatives and concludes that

$$v \in C^{2,\alpha}(Q_1) \quad \text{for some } \alpha > 0.$$

- **Bootstrapping to C^∞.** Once $C^{2,\alpha}$-regularity is available, the equation (5.5) can be differentiated arbitrarily many times. Each derivative $D^\ell v$ satisfies a linear second-order equation whose coefficients are combinations of $x_1 v_{11}$, $x_2 v_{22}$ and their derivatives. Using again the same coordinate changes as in the analysis of v_{12}, Huang shows that these linear operators are uniformly elliptic in suitable Euclidean coordinates and that their coefficients are C^α up to the origin. Standard Schauder estimates then imply that
$$v \in C^{k,\alpha_k}(Q_1) \quad \text{for all } k \geq 2,$$
with control of weighted quantities such as $x_1 D^{k+1} v$ and $x_2 D^{k+1} v$. This yields $v \in C^\infty(\overline{Q}_1)$, and therefore
$$u(x) - x_1 \log x_1 - x_2 \log x_2 \in C^\infty(\overline{Q}_1)$$
in the vertex coordinates.

Combining this corner analysis with Rubin's global existence and edge regularity, one finally obtains Huang's theorem: for any strictly positive $h \in C^\infty(\overline{P})$ satisfying the natural compatibility conditions, and for any prescribed vertex values, there exists a unique convex solution u of (5.4) on P such that
$$u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}).$$
In other words, in the polygonal case the Guillemin boundary problem is completely well posed and enjoys full boundary regularity in the natural function space dictated by the toric geometry.

5.5. **Huang–Shen: high-dimensional simple polytopes.** Huang and Shen extended the theory of the Guillemin boundary problem to arbitrary dimension [HS]. Let $P \subset \mathbb{R}^n$ be a simple convex polytope,
$$P = \bigcap_{k=1}^{N} \{\ell_k(x) > 0\}, \qquad \ell_k \text{ affine},$$
and let $h \in C^\infty(\overline{P})$ be strictly positive. They consider the boundary value problem
$$\det D^2 u = \frac{h(x)}{\prod_{k=1}^{N} \ell_k(x)} \quad \text{in } P, \tag{5.6}$$
together with the Guillemin boundary condition
$$u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}), \tag{5.7}$$
and ask for convex solutions u.

They first show that the geometric assumption that P is simple is not merely natural but necessary: if more than n facets meet at a vertex then (5.6)-(5.7) cannot hold with h smooth and positive [HS, Prop. 2.1]. Next, they identify a necessary compatibility condition at each vertex. Let p_i be a vertex where exactly n facets $\ell_{i_1}, \dots, \ell_{i_n}$ vanish and all other $\ell_k(p_i)$ are nonzero. Then any solution of (5.6)-(5.7) must satisfy

$$h(p_i) = \prod_{\ell_k(p_i)\neq 0} \ell_k(p_i) \left[\det(D\ell_{i_1}, \dots, D\ell_{i_n})\right]^2. \tag{5.8}$$

Conversely, this family of algebraic conditions at the vertices is sufficient. More precisely, Huang and Shen prove:

If P is a simple convex polytope and $h \in C^\infty(\overline{P})$ is strictly positive and satisfies the vertex compatibility conditions (5.8)*, then for any prescribed real numbers $\alpha_1, \dots, \alpha_{N_b}$ assigned to the vertices $p_1, \dots, p_{N_b}$ of P there exists a unique convex solution u of* (5.6)-(5.7) *such that $u(p_i) = \alpha_i$ for all i, and*

$$f(x) := u(x) - \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}).$$

In particular, the singularity of u along ∂P in any dimension is again exactly the universal sum $\sum \ell_k \log \ell_k$.

A characteristic feature of their work is that boundary values can only be prescribed at the vertices. Lemma 2.2 in [HS] shows that the restriction of u to any face of codimension k solves a lower-dimensional Guillemin boundary problem on that face. Thus the boundary data on higher-dimensional faces are determined inductively from the vertex values via lower-dimensional Monge-Ampère equations, and the solvability of the n-dimensional problem depends on the solvability in all smaller dimensions.

The proof is technically involved and combines several new analytic ingredients:

- **Existence and interior regularity.** Using classical results on the Dirichlet problem for the Monge-Ampère equation, they first solve a Dirichlet problem on P with boundary data compatible with the Guillemin ansatz and obtain an Alexandrov solution $u \in C(\overline{P})$ [HS, Thm. 2.1]. A careful barrier argument near facets and a strict convexity argument inspired by Pogorelov's example then yield global strict convexity and interior C^∞ regularity of u [HS, Thm. 2.2].
- **Regularity up to facets (codimension one).** Near an $(n-1)$-dimensional face, after an affine change of coordinates one is reduced to the model situation

  $$\det D^2 u = \frac{h(x)}{x_1} \quad \text{in } Q_3 = (0,3) \times (-3,3)^{n-1},$$

  with $u(0, x'')$ a smooth, uniformly convex solution of a non-singular Monge-Ampère equation in $x'' \in \mathbb{R}^{n-1}$ [HS, (1.12)-(1.13)]. Writing

  $$v(x) := u(x) - x_1 \log x_1,$$

they prove first a Lipschitz estimate $|v(x)-v(0,x'')| \leq Cx_1$ up to the boundary and then a weighted $C^{1,1}$ estimate of the form

$$x_1|v_{11}| + \sqrt{x_1}\sum_{i\geq 2}|v_{1i}| + \sum_{i,j\geq 2}|v_{ij}| \leq C$$

near the facet [HS, Lem. 3.1-3.2]. These estimates allow a partial Legendre transform in the tangential variables, turning the problem into a quasilinear elliptic equation with bounded measurable coefficients. A weighted $C^{2,\alpha}$ estimate for v is then obtained, and a bootstrap argument using linear elliptic estimates yields C^∞ regularity of v up to the facet [HS, Thms. 3.1-3.2].

- **Model problems on orthants and Lipschitz regularity at corners.** For a fixed $2 \leq k \leq n$, in a neighbourhood of a face of codimension k one reduces to equations of the type

$$\det D^2 u = \frac{h(x)}{x_1 \cdots x_k} \quad \text{in } Q_3 = (0,3)^k \times (-3,3)^{n-k},$$

with appropriate boundary conditions on the coordinate hyperplanes [HS, (1.14)-(1.16)]. Setting

$$v(x) := u(x) - \sum_{i=1}^{k} x_i \log x_i,$$

they first obtain a weak asymptotic estimate

$$|v(x) - F(x)| \leq C\,(x_1 \cdots x_k)^{1/k} + o(|x'|),$$

where F is a smooth extension of the boundary data [HS, (1.17)-(1.18)]. This implies that v is Lipschitz up to $(n-k)$-dimensional faces.

- **Liouville theorem and refined asymptotics.** A key step is a Liouville-type theorem on the infinite orthant $(\mathbb{R}_+)^k \times \mathbb{R}^{n-k}$, which characterizes global Alexandrov solutions of the model equation

$$\det D^2 u = \frac{1}{x_1 \cdots x_k}$$

with prescribed boundary values and controlled growth [HS, Thm. 1.2]. Applying a blow-up argument and this Liouville theorem, they improve the weak asymptotic estimate to the sharp bound

$$|v(x) - F(x)| \leq C\, x_1 \cdots x_k,$$

which is precisely the order of the singular right-hand side.

- **Induction on the codimension and higher regularity.** The refined asymptotic control is combined with elliptic estimates for suitable derivatives of v (in particular $v_{1\cdots k}$) after carefully chosen coordinate changes. This allows one to reduce boundary regularity near a face of codimension k to interior regularity of an auxiliary equation. An induction on k then yields C^∞ regularity of v up to faces of all codimensions [HS, Sec. 4].

One of the striking features of Huang-Shen's work is that the combinatorics of the simple polytope P are fully reflected in the analytic structure: the compatibility conditions on h are purely algebraic at the vertices, and the regularity theory is organized inductively over faces of increasing codimension. Their result shows that the Guillemin boundary problem for the Monge-Ampère equation is analytically natural in any dimension: the only singularities compatible with global smoothness are exactly the logarithmic terms $\sum \ell_k \log \ell_k$ dictated by the geometry of toric divisors, and after subtracting this universal singularity the solution is smooth on $\overline{P}$.

5.6. **Boundary Schauder estimates: Bayrami-Seyyedali-Talebi.** While the works of Rubin, Huang, and Huang-Shen assume a smooth positive function h, it is important for applications to understand what happens when the right-hand side is only Hölder continuous. Very recent work of Bayrami-Aminlouee, Seyyedali, and Talebi addresses this question in the polygonal case $n = 2$ [BST].

Let $P \subset \mathbb{R}^2$ be a convex polygon, and consider

$$\det D^2 u = \frac{H(x)}{\prod_{k=1}^N \ell_k(x)}, \qquad H \in C^{0,\alpha}(\overline{P}),\ H > 0,$$

with the Guillemin boundary condition

$$u(x) - \sum_{k=1}^N \ell_k(x) \log \ell_k(x) \in C^0(\overline{P}).$$

A central role is played by the reference potential

$$u_0(x) = \sum_{k=1}^N \ell_k(x) \log \ell_k(x),$$

whose Hessian defines a canonical Riemannian metric

$$g_0 = \sum_{i,j} (u_0)_{ij}(x)\, dx_i dx_j$$

on P. This metric captures the degeneracy of the equation near the boundary: g_0 is asymptotically hyperbolic along edges and has a product-type structure near vertices.

Bayrami-Seyyedali-Talebi show that for any solution u of the above problem one has:

- a global equivalence of Hessians
$$C^{-1} D^2 u_0 \le D^2 u \le C D^2 u_0 \quad \text{in } P,$$
for a constant C depending only on geometric data and the $C^{0,\alpha}$-norm of H;
- a Schauder-type estimate
$$\|u - u_0\|_{C^{2,\alpha}_{g_0}(P)} \le C,$$
where $C^{2,\alpha}_{g_0}$ is a weighted Hölder space defined using the distance function of the metric g_0 [BST].

These estimates are optimal in the natural geometric scale determined by g_0. In particular, they imply that the singular geometry encoded by u_0 is robust under perturbations of the right-hand side: even if H is only Hölder continuous, the solution u remains uniformly comparable to u_0 at the level of second derivatives, and the difference $u - u_0$ is $C^{2,\alpha}$ in the weighted sense.

The proof combines the global geometry of (P, g_0) with local perturbative arguments around model solutions. Donaldson's interior estimates for the Abreu equation and the linearized Monge-Ampère theory developed by Chen, Han, Li, and Sheng provide important ingredients for controlling the behaviour of D^2u in the interior of P [Do2, CHLS]. Near the boundary, the authors exploit the explicit structure of u_0 to construct barriers and to transfer classical Schauder estimates to the weighted spaces $C^{2,\alpha}_{g_0}$.

These results dovetail perfectly with Huang's smooth theory in the case $H \in C^\infty$ [Hu]. Together, they yield a flexible and robust regularity theory for the Monge-Ampère equation with Guillemin boundary conditions on polygons, which is well suited for applications to the Abreu equation and related fourth-order problems. They are philosophically close to the boundary regularity theory for Monge-Ampère and affine maximal surface equations developed by Trudinger-Wang [TW5] and to the general Monge-Ampère-type boundary problems studied in [TW6, TW4].

6. Further PDE theory for the Abreu equation

6.1. **Degenerate boundary conditions and complex tori.** Chen, Li, and Sheng initiated a systematic PDE study of the Abreu equation with degenerate boundary conditions [CLS1]. They consider convex solutions u of

$$-\sum_{i,j=1}^{n} \partial_{ij} u^{ij} = K(x)$$

on a bounded, strictly convex domain $\Omega \subset \mathbb{R}^n$, subject to the boundary condition

$$u(x) \to +\infty \quad \text{as } x \to \partial\Omega$$

in such a way that the Legendre transform

$$v(y) = \sup_{x \in \Omega} \{\langle x, y \rangle - u(x)\}$$

is defined on all of $\mathbb{R}^n$ and has controlled growth [CLS1]. This setup is tailored so that, after identifying $\mathbb{R}^n$ with the universal cover of $(\mathbb{C}^*)^n$, the function v descends to a global Kähler potential on a flat complex torus $(\mathbb{C}^*)^n/\Lambda$ and the metric associated to u solves a constant scalar curvature equation on that torus [CLS2].

The first paper [CLS1] analyzes the Dirichlet-type problem with such degenerate boundary data on Ω. A key point is to reinterpret the Abreu equation via the Legendre transform: the equation for u on Ω becomes, for v, an equation on the dual domain with a right-hand side involving K composed with the gradient map and the determinant of the Hessian. This dual description allows one to exploit techniques from the theory of the real Monge-Ampère equation-such as Aleksandrov solutions, barrier constructions, and Caffarelli-type interior estimates-for the pair (u, v) [CLS1]. Many of these techniques are closely related to those used in the study of the second boundary value problem for Monge-Ampère type equations in optimal transport, developed by Urbas and by Ma-Trudinger-Wang and Trudinger-Wang [MTW, TW6].

In the companion work [CLS2], Chen-Li-Sheng use this degenerating boundary condition to construct many cscK metrics on complex tori. Starting from a flat background torus, they prescribe a scalar curvature function K satisfying natural integral constraints and solve the Abreu equation with blow-up boundary conditions on a fundamental convex domain Ω in $\mathbb{R}^n$. The Legendre transform then yields a smooth Kähler potential on the torus whose associated Kähler metric has scalar curvature K [CLS2]. They also establish uniqueness and regularity results, showing that the degenerating boundary condition is compatible with a robust elliptic theory.

Analytically, the main features of their approach include:

- careful control of the growth of u near $\partial\Omega$, guaranteeing that the gradient map ∇u is a diffeomorphism from Ω onto $\mathbb{R}^n$;
- global determinant bounds for $\det D^2u$ via maximum principles and barrier functions adapted to the blow-up behaviour;
- application of linearized Monge-Ampère estimates to obtain $C^{2,\alpha}$ and higher regularity in the interior, followed by a delicate analysis of the degenerating boundary asymptotics.

These results show that the Abreu equation can be handled in settings quite different from the Guillemin boundary condition on Delzant polytopes, and they provide an important class of non-toric examples of cscK metrics constructed by purely PDE methods [CLS1, CLS2].

6.2. **Interior estimates in general dimension.** Donaldson's interior estimates for Abreu's equation in real dimension two [Do2] were extended to arbitrary dimension by Chen, Han, Li, and Sheng [CHLS]. They study convex solutions u of

$$-\sum_{i,j=1}^{n} \partial_{ij} u^{ij} = K(x)$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, under the assumptions that D^2u is uniformly positive definite and that the pair (Ω, K) satisfies a quantitative stability condition modeled on Donaldson's positivity condition for the functional L_A [Do1, CHLS].

A first step is to obtain two-sided bounds for the Monge-Ampère measure $\det D^2u$ on compact subsets $\Omega' \Subset \Omega$. Using a refined maximum principle and suitable auxiliary functions, they prove that if K is bounded and the stability condition holds, then

$$c^{-1} \leq \det D^2u \leq c$$

on Ω', with c depending only on geometric and stability data [CHLS]. These determinant bounds imply uniform ellipticity for the linearized operator

$$\mathcal{L}_u v := u^{ij}\partial_{ij}v$$

in the sense of the linearized Monge-Ampère theory of Caffarelli-Gutiérrez [CHLS]. One can then invoke interior Harnack inequalities and $C^{2,\alpha}$ estimates for solutions of $\mathcal{L}_u v = f$, and bootstrap to obtain higher regularity.

More precisely, Chen-Han-Li-Sheng show that for any $\Omega' \Subset \Omega$ and any $k \geq 0$, there exist constants $C_{k,\alpha}$ such that

$$\|u\|_{C^{k,\alpha}(\Omega')} \leq C_{k,\alpha}$$

provided that $\|K\|_{C^{k-2,\alpha}(\Omega)}$ and the stability constants are controlled [CHLS]. The proof combines:

- Legendre transform techniques to exploit the divergence form of Abreu's operator;
- Caffarelli-type regularity for the linearized Monge-Ampère equation, applied to u and to various derivatives of u;
- iteration of Schauder estimates using the structure of K and the stability condition.

These interior estimates are flexible enough to be applied in the toric setting, in the degenerate boundary problems of Chen-Li-Sheng, and in the generalized Abreu equations discussed below [CLS1, LLS2].

6.3. **Generalized Abreu equations.** In the study of homogeneous toric bundles and more general fibrations, the scalar curvature equation in toric coordinates often takes the form of a *generalized Abreu equation*

$$-\sum_{i,j=1}^{n} \frac{1}{\varrho(x)}\,\partial_{ij}\big(\varrho(x)\,u^{ij}\big) = A(x) \tag{6.1}$$

on a polytope $P \subset \mathbb{R}^n$, where $\varrho(x)$ is a smooth positive weight function encoding the contribution of the fibre geometry and $A(x)$ is a prescribed smooth function related to the desired scalar curvature profile [LLS1, LLS2, Ny]. When $\varrho \equiv 1$ this reduces to the usual Abreu equation.

Li, Lian, and Sheng developed a systematic interior regularity theory for (6.1), parallel to the unweighted case [LLS1, LLS2]. They assume an appropriate notion of *uniform K-stability* for the triple (P, ϱ, A), which can be formulated as positivity properties of a weighted linear functional

$$L_{A,\varrho}(u) = \int_{\partial P} u \,\mathrm{d}\sigma_\varrho - \int_P A(x)\,u(x)\,\varrho(x)\,\mathrm{d}x$$

on convex functions; here σ_ϱ is a boundary measure depending on ϱ [LLS1]. Under this stability assumption and uniform convexity of u, they establish interior determinant bounds for $\det D^2u$ and then use weighted linearized Monge-Ampère estimates to obtain $C^{k,\alpha}$ control for u on compact subsets of P [LLS2]. The underlying Monge-Ampère theory fits naturally into the general framework of Monge-Ampère type equations surveyed by Trudinger-Wang [TW4].

Geometrically, such generalized equations arise when one looks for extremal metrics on homogeneous toric bundles or more general fibrations whose base is a toric manifold and whose fibres carry canonical metrics [Ny]. In these situations the scalar curvature of the total space can be expressed in terms of data on the base polytope, with the weight ϱ encapsulating the volume density of the fibres. The analytic theory of (6.1) thus plays a key role in constructing extremal metrics in non-product situations, and it illustrates how the methods developed for the standard Abreu equation extend to more intricate geometric settings [LLS1, LLS2, Ny].

6.4. **Extremal metrics and K-stability via PDE.** Building on Donaldson's variational framework [Do1, Do3, Do4], Chen, Li, and Sheng developed a PDE-variational approach to extremal metrics on toric surfaces [CLS3, CLS4]. Instead of relying primarily on a continuity method, they work directly with the modified Mabuchi K-energy functional on the space of symplectic potentials and relate its coercivity to a notion of *uniform* relative K-stability.

For a Delzant polygon P with boundary measure σ and a smooth function A on P representing the desired scalar curvature, they study the functional

$$F_A(u) = -\int_P \log\det D^2u \; \mathrm{d}x + \int_{\partial P} u \; \mathrm{d}\sigma - \int_P A(x)\, u(x) \; \mathrm{d}x,$$

which is the toric Mabuchi energy up to normalization [Do1, CLS3]. Under an appropriate normalization of u (for instance, fixing its value at a point and its average over P), uniform relative K-stability can be expressed as the inequality

$$L_A(f) \geq \delta \, \|f\|_{\mathrm{norm}}$$

for all rational convex piecewise linear functions f orthogonal to the extremal affine functions, for some $\delta > 0$ and a suitable norm [CLS4]. This quantitative positivity implies a coercivity inequality

$$F_A(u) \geq \delta' \, \|u\|_{\mathrm{norm}} - C$$

for all u in the Guillemin class, which in turn yields the existence of minimizing sequences with good compactness properties [CLS3, CLS4].

Chen-Li-Sheng then study weak limits of minimizing sequences and show that any minimizer in a suitable completion of the space of convex functions solves the Abreu equation in an Alexandrov or distributional sense [CLS3]. Using their interior estimates and regularity theory, they upgrade such weak solutions to smooth extremal metrics on the toric surface. This yields a PDE proof of Donaldson's existence results for extremal toric metrics and clarifies the role of uniform K-stability in ensuring quantitative control over minimizers [CLS3, CLS4].

Conceptually, this variational approach is very close in spirit to the general theory developed later by Chen and Cheng for cscK metrics on arbitrary Kähler manifolds, where properness of the K-energy in a suitable metric topology is shown to be equivalent to the existence of a smooth minimizer [CC2]. The toric setting provides a concrete model in which all the objects involved can be written explicitly in terms of convex functions on a polytope.

6.5. **Chen-Cheng's existence theory for cscK metrics.** The general Yau-Tian-Donaldson picture suggests that the existence of cscK metrics on a polarized manifold should be equivalent to a suitable stability or properness condition for energy functionals on the space of Kähler potentials. Chen and Cheng made decisive progress on the analytic side of this conjecture in their trilogy on the cscK equation [CC1, CC2, CC3].

Let (X, ω) be a compact Kähler manifold, and write $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi$ for a Kähler metric in the same cohomology class. The cscK equation asks for

$$S(\omega_\varphi) = \underline{S},$$

where $S(\omega_\varphi)$ is the scalar curvature and $\underline{S}$ is its topological average. In complex coordinates this is a fully nonlinear fourth-order equation in φ, but it can be viewed as a second-order equation for the Kähler metric coupled with its Ricci potential.

In the first paper, Chen and Cheng establish a complete *a priori* estimate theory for the cscK equation: they show that a uniform C^0 bound on the Kähler potential φ implies uniform bounds for all derivatives of φ [CC1]. More precisely, under mild assumptions on the background geometry, they derive $C^{2,\alpha}$-bounds from a C^0-bound via delicate estimates for the complex Monge-Ampère part of the equation and the linearized scalar curvature operator, and then bootstrap to obtain $C^{k,\alpha}$-bounds for all k. This provides the analytic backbone needed to run continuity methods or variational arguments, reducing existence questions to the problem of obtaining uniform C^0 control.

The second paper identifies a natural properness condition that guarantees such control. Working on the metric completion of the space of Kähler potentials with respect to the L^1 Mabuchi (or d_1) distance, they prove that the Mabuchi K-energy is proper if and only if it admits a minimizer, and that any minimizer corresponds to a smooth cscK metric [CC2]. In particular, properness of the K-energy implies the existence of a cscK metric, and conversely the existence of a cscK metric forces a suitable properness inequality. This confirms, in analytic form, a conjecture of Donaldson relating geodesic stability and the existence of cscK metrics [Do1, CC2].

In the third paper, Chen and Cheng extend their theory to the case where the automorphism group $\mathrm{Aut}^0(X)$ is nontrivial [CC3]. They introduce a modified K-energy functional and a corresponding notion of properness on the quotient by $\mathrm{Aut}^0(X)$, and prove that properness of this modified functional is equivalent to the existence of an extremal Kähler metric. This yields an analytic existence theory for extremal metrics that parallels the cscK case and is compatible with the Futaki invariant and extremal vector fields.

Although these results are formulated in the general Kähler setting rather than in toric coordinates, they have important conceptual implications for the Abreu equation. On a toric manifold, the cscK (or extremal) equation written in complex coordinates is equivalent to the Abreu equation for the symplectic potential with Guillemin boundary conditions. Chen-Cheng's a priori estimates show that once one can control the Kähler potential in C^0, all higher-order regularity follows, while Donaldson's work and the Guillemin boundary theory provide a complementary real-variable framework adapted to the polytope [Do2, Hu, HS, BST]. In particular, the toric results discussed in Section 4 and Section 5 can be viewed as a model case where the general analytic theory of Chen and Cheng can be made completely explicit.

Moreover, when combined with algebro-geometric characterizations of properness in terms of K-stability, Chen-Cheng's work furnishes the analytic half of the YTD conjecture for general polarized manifolds. In the Fano case, this picture is consistent with (and partly underlies) the proofs of the YTD conjecture for Kähler-Einstein metrics due to Tian [Ti2] and to Chen-Donaldson-Sun [CDS1, CDS2, CDS3].

6.6. **Other boundary value problems and affine geometry.** The Abreu equation belongs to a broader class of fourth-order equations built from the Monge-Ampère operator and its linearization. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth strictly convex function and set

$$\phi = \det D^2u, \qquad U^{ij} = \operatorname{cof}(D^2u)_{ij}.$$

The linearized Monge-Ampère operator at u is

$$L_u w := U^{ij} w_{ij}.$$

In many geometric problems one studies equations of the form

$$L_u w = F(x, u, \nabla u),$$

where w is a nonlinear function of ϕ (or more generally of D^2u) determined by the underlying geometry. The Abreu equation fits into this framework insofar as it can be seen as a fourth-order equation governed by the same tensor U^{ij}, but written in divergence form as

$$-\partial_i \partial_j u^{ij} = A(x),$$

where u^{ij} is the inverse matrix of u_{ij}. Thus, from the analytic point of view, Abreu's equation is tightly linked to affine differential geometry and to the theory of prescribed affine curvatures developed by Trudinger, Wang and others [TW1, TW2, TW3, TW4].

Affine maximal hypersurfaces and prescribed affine mean curvature. Consider the graph

$$M = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}$$

of a convex function u over a bounded, uniformly convex domain $\Omega \subset \mathbb{R}^n$. In affine differential geometry, a fundamental role is played by the *affine area functional*

$$\mathcal{A}(u) = \int_\Omega \phi^{\frac{1}{n+2}} \, \mathrm{d}x,$$

which is invariant under volume-preserving affine transformations of $\mathbb{R}^{n+1}$. The Euler-Lagrange equation associated with critical points of $\mathcal{A}$ is the *affine maximal hypersurface equation*

$$L_u w = 0, \qquad w = \phi^{-\frac{n+1}{n+2}},$$

a fourth-order, fully nonlinear, affinely invariant PDE [TW1, TW2]. More generally, if one prescribes the affine mean curvature of M, the resulting equation is of the form

$$L_u w = f(x), \qquad w = \phi^{-\frac{n+1}{n+2}},$$

for a given function f; this is often referred to as the *prescribed affine mean curvature equation* [TW3, TW4]. These equations provide prototypical examples of fourth-order Monge-Ampère type equations and are structurally very close to Abreu's equation, in that the same tensor U^{ij} governs the principal part.

The analytic theory of affine maximal and prescribed affine mean curvature equations was developed in a series of works of Trudinger and Wang, in which they introduced the notion of Hessian measures, developed a priori estimates adapted to the affine structure, and established Bernstein-type theorems, existence results, and fine regularity for various boundary value problems [TW1, TW2, TW3, TW5].

Second boundary value problems. For the Monge-Ampère equation

$$\det D^2 u = g(x) \quad \text{in } \Omega,$$

a classical *second boundary value problem* (in the sense of Brenier, Caffarelli, Ma–Trudinger–Wang, Trudinger–Wang, etc.) consists in prescribing both the boundary values of the potential and the image of the gradient map:

$$u = \varphi \quad \text{on } \partial\Omega, \qquad \nabla u(\Omega) = \Omega^*,$$

for a given uniformly convex target domain $\Omega^* \subset \mathbb{R}^n$. Equivalently, one prescribes the transport map ∇u (or the associated Monge-Ampère measure of Ω) rather than just the density g [MTW, TW6, TW4]. In the optimal transport interpretation, this corresponds to transporting a source measure supported on Ω to a target measure supported on Ω^* with u playing the role of the transport potential.

For fourth-order equations of affine type, such as the general prescribed affine mean curvature equation

$$U^{ij} w_{ij} = F(x, u, \nabla u) \quad \text{in } \Omega, \qquad w = G'\big(\det D^2 u\big), \tag{6.2}$$

the natural "second boundary value problem" couples two sets of boundary data, reflecting the fact that (6.2) can be viewed as a nonlinear system for the pair (u, w). One typically prescribes

$$u = \varphi \quad \text{on } \partial\Omega, \qquad w = \psi \quad \text{on } \partial\Omega,$$

where φ and ψ are given functions determined by the geometric problem (affine maximal hypersurfaces, prescribed affine mean curvature, Abreu-type equations, geometric optics, etc.). In particular, for the affine maximal surface equation and more general prescribed affine mean curvature equations, this formulation arises as the natural Euler–Lagrange boundary condition for variational problems involving the affine area or affine mean curvature functionals [TW1, TW2, TW3, CW, Le1, Le2, Le7, KLWZ].

The analysis of such fourth-order second boundary value problems relies heavily on the underlying Monge-Ampère structure. *A priori* estimates and global regularity for u and w are obtained by combining the interior theory for Monge-Ampère and linearized Monge-Ampère equations with delicate boundary estimates for the associated second-order problems [TW5, Le7]. Results of this type provide a blueprint for treating boundary value problems for Abreu's equation and other fourth-order Monge-Ampère type equations.

To summarize, the Abreu equation sits at the intersection of several rich theories: Kähler geometry and canonical metrics, affine differential geometry, optimal transport, and the PDE theory of Monge-Ampère and linearized Monge-Ampère equations. The analytic techniques developed in the affine setting by Trudinger, Wang and coauthors, and the more recent contributions of Le [Le7, MTL] and others to boundary regularity and linearized Monge-Ampère theory, provide a powerful toolkit that extends well beyond the toric context and is likely to be applicable to a wide range of geometric fourth–order problems.

7. Connections and further discussions

7.1. **Guillemin boundary problem as a model for singular boundary behaviour.** The Guillemin boundary problem for the Monge-Ampère equation provides a prototypical setting where:

- the domain has corners and a stratified boundary (faces, edges, vertices, and higher-codimension strata in general dimensions);
- the equation is singular or degenerate at the boundary, with a right-hand side that blows up like the reciprocal of a product of the defining functions of the facets;
- solutions are nevertheless smooth up to the boundary after subtracting a universal singular part determined purely by the polytope.

Concretely, on a simple polytope $P \subset \mathbb{R}^n$, one considers

$$\det D^2 u = \frac{h(x)}{\prod_{k=1}^N \ell_k(x)}, \qquad u(x) - \sum_{k=1}^N \ell_k(x) \log \ell_k(x) \in C^\infty(\overline{P}),$$

with $h > 0$ smooth; cf. Definition (5.3). The work of Rubin, Huang, Huang-Shen, and Bayrami-Seyyedali-Talebi [Ru, Hu, HS, BST] shows that, under natural compatibility conditions and appropriate regularity hypotheses on h, this problem is well-posed and that

$$u(x) = \sum_{k=1}^{N} \ell_k(x) \log \ell_k(x) + f(x)$$

with f smooth up to the boundary (or in suitable weighted Hölder classes when h is only Hölder). In other words, the singular behaviour of u near ∂P is completely captured by the model potential $u_0 = \sum_k \ell_k \log \ell_k$.

This makes the Guillemin boundary problem an attractive model for other fully nonlinear elliptic equations with comparable boundary singularities. For instance:

- k-Hessian equations of the form $\sigma_k(D^2u) = f(x) \prod_k \ell_k(x)^{-1}$ on polytopes, where one expects a similar decomposition into a universal singular part plus a smooth remainder;
- geometric equations arising from curvature-prescription problems (e.g., prescribed Gauss curvature or affine mean curvature) on domains with corners, where the natural boundary data enforce singular metrics along the boundary;
- more general elliptic operators $F(D^2u, x)$ that are asymptotically equivalent, near ∂P, to a Monge-Ampère-type operator with a right-hand side comparable to $\prod_k \ell_k(x)^{-1}$.

A natural family of questions is:

- to classify which operators F admit a canonical "Guillemin-type" singular part (depending only on the combinatorics of P and not on the solution itself);
- to identify the minimal regularity assumptions on h (or on the data in more general equations) that guarantee optimal regularity for the smooth part f of the solution (in standard or weighted Hölder spaces);
- to understand to what extent the polyhedral geometry—angles at vertices, dihedral angles along edges, combinatorics of faces—influences the mapping properties (Fredholmness, Schauder estimates) of the linearized operator around the model solutions.

Progress on these questions, even for variants of the Monge-Ampère equation close to the Guillemin model, would enlarge the range of singular geometric problems amenable to the techniques developed in [Ru, Hu, HS, BST].

7.2. **Quantitative stability thresholds and estimates.** Donaldson's toric work [Do1, Do4] suggests that *uniform* K-stability, rather than mere stability, should correspond to quantitative coercivity of the K-energy functional, and hence to strong *a priori* estimates for solutions of the Abreu equation. In the toric setting, K-stability can be expressed in terms of positivity of the linear functional

$$L_A(f) = \int_{\partial P} f \, \mathrm{d}\sigma - \int_P A(x) f(x) \, \mathrm{d}x$$

on rational convex piecewise linear functions f, while uniform K-stability strengthens this to an inequality of the form

$$L_A(f) \geq \delta \, \|f\|_{\mathrm{norm}}$$

for some fixed $\delta > 0$ and a suitable norm on the space of normalized convex functions [CLS4, LLS1].

This kind of quantitative positivity has several expected analytic consequences:

- a uniform lower bound for the toric Mabuchi functional F_A, hence precompactness for minimizing sequences in appropriate topologies;
- uniform control of oscillation and higher derivatives for symplectic potentials u solving the Abreu equation, via variational and blow-up arguments [CLS3, LLS2];
- stability of solutions under small perturbations of the data (P, A, σ), in the sense that the solution map is continuous with respect to these parameters.

While Chen–Li–Sheng and Li–Lian–Sheng [CLS4, CLS3, LLS1, LLS2] have begun to make these expectations precise in the toric and generalized toric settings, many questions remain open. For instance:

- Can one give explicit estimates linking the stability constant δ in $L_A(f) \geq \delta\|f\|$ to ellipticity constants for D^2u (i.e., two-sided bounds on $\det D^2u$ and its eigenvalues) and to quantitative interior and boundary $C^{k,\alpha}$ estimates?
- Is there a sharp threshold phenomenon: as $\delta \to 0$, do solutions of the Abreu equation develop geometric degenerations (e.g., bubbling or collapsing), and can these degenerations be classified in terms of limiting destabilizing test configurations?
- How do these quantitative toric estimates fit into the broader non-toric picture, where uniform K-stability and properness of the K-energy are characterized in terms of valuations and non-Archimedean functionals [CC2, Ti2]?

Developing a fully quantitative theory in the toric case would not only refine the existing existence theorems, but also provide a testing ground for similar questions in the general cscK problem, where analytic and algebro-geometric notions of stability are still being matched at a quantitative level [CC1, CC2, CC3, CDS1, CDS2, CDS3].

7.3. **Beyond toric geometry.** Although the Abreu equation and Guillemin boundary conditions arise canonically in toric Kähler geometry, similar structures appear in a variety of other settings.

Homogeneous toric bundles and projective bundles. For certain fibrations whose base is a toric manifold and whose fibres carry homogeneous or canonical metrics, the scalar curvature on the total space can be expressed in terms of a generalized Abreu equation on the base polytope with an additional weight $\varrho(x)$ encoding the fibre geometry [LLS1, LLS2, Ny]. This yields a rich family of examples where polyhedral and convex-analytic techniques still apply, but the effective equation is more complicated than in the pure toric case. Understanding stability and properness in this weighted setting and relating them to the existence of extremal metrics on such bundles remains an active area of research [Ny, LLS1].

Manifolds with large symmetry and complexity-one actions. Beyond toric varieties (complexity zero), one can consider Kähler manifolds with an effective Hamiltonian action of a torus of intermediate dimension, such as complexity-one T-varieties. In favourable situations, one still has a combinatorial description involving polyhedral divisors or moment graphs, and the scalar curvature equation can often be reduced to PDEs on lower-dimensional polyhedral spaces coupled with ODEs along the orbits [Ny]. Extending Abreu-type techniques to these settings, including the development of appropriate boundary conditions and stability notions, is an attractive but challenging direction.

Affine differential geometry and optimal transport. As discussed in Section 6, equations closely related to Abreu's arise in affine differential geometry, for instance in the prescribed affine mean curvature problem and in the study of affine maximal hypersurfaces [CHLS, TW1, TW2, TW3]. On the other hand, the Monge-Ampère equation and its linearization are central in optimal transport theory and in variational problems with convexity constraints. Recent work shows that Abreu-type fourth-order PDE can be used to approximate variational problems with convexity constraints, providing an analytic tool to enforce convexity in the calculus of variations [CR, Le3, Le4, Le5, Le6, LZ]. In all these contexts, the ideas developed for the Abreu equation, such as Legendre transforms, linearized Monge-Ampère theory, and boundary regularity on domains are likely to play a significant role.

In summary, the Abreu equation and the Guillemin boundary problem lie at a meeting point of several important areas of mathematics: Kähler geometry, convex analysis, and nonlinear PDEs. Their study shows how questions about special metrics on spaces with symmetry can be expressed through concrete and difficult analysis on polyhedral domains. While much has been understood in the toric setting, the ideas and methods developed here are not limited to that case. The way singularities are handled, the role of stability conditions, and the techniques for obtaining estimates naturally suggest broader applications. Looking ahead, progress in any one of these connected fields will likely inspire and reinforce advances in the others.

Acknowledgements

This survey is purely expository, and any errors or omissions are the sole responsibility of the author. The reader is encouraged to consult the original articles cited herein for full details. The author would like to express his gratitude to his PhD advisor, Prof. Bin Zhou, for constant encouragement, and to his postdoctoral mentor, Prof. Antonio De Rosa, for his generous support.

The author was funded by the European Union: the European Research Council (ERC), through StG "ANGEVA", project number: 101076411. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

References

[Ab] M. Abreu, Kähler geometry of toric varieties and extremal metrics, *Int. J. Math.* **9** (1998), no. 6, 641-651.

[ACG] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon, Hamiltonian 2-forms in Kähler geometry. I. General theory, *J. Differential Geom.* **73** (2006), no. 3, 359-412.

[At] M. F. Atiyah, Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.

[BST] M. Bayrami-Aminlouee, R. Seyyedali, and M. Talebi, Boundary regularity for the Monge-Ampère equation with Guillemin boundary conditions in polygons, preprint (2025), available at arXiv.

[Ca] E. Calabi, Extremal Kähler metrics, in *Seminar on Differential Geometry*, Annals of Math. Studies **102**, Princeton Univ. Press, 1982, 259-290.

[CR] G. Carlier and T. Radice, Approximation of variational problems with a convexity constraint by PDEs of Abreu type, *Calc. Var. Partial Differential Equations* **58** (2019), no. 5, Paper No. 170, 13 pp.

[CW] A. Chau and B. Weinkove, Monge-Ampère functionals and the second boundary value problem, *Math. Res. Lett.* **22** (2015), no. 4, 1005-1022.

[CC1] X. X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (I)-a priori estimates, *J. Amer. Math. Soc.* **34** (2021), no. 4, 909-936.

[CC2] X. X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (II)-existence results, *J. Amer. Math. Soc.* **34** (2021), no. 4, 937-1009.

[CC3] X. X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (III)-general automorphism group, preprint (2018), arXiv:1801.05907.

[CDS1] X. Chen, S. K. Donaldson, and S. Sun, Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities, *J. Amer. Math. Soc.* **28** (2015), no. 1, 183-197.

[CDS2] X. Chen, S. K. Donaldson, and S. Sun, Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π, *J. Amer. Math. Soc.* **28** (2015), no. 1, 199-234.

[CDS3] X. Chen, S. K. Donaldson, and S. Sun, Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the proof, *J. Amer. Math. Soc.* **28** (2015), no. 1, 235-278.

[CHLS] B. Chen, Q. Han, A.-M. Li, and L. Sheng, Interior estimates for the n-dimensional Abreu's equation, *Adv. Math.* **251** (2014), 35-46.

[CLS1] B. Chen, A.-M. Li, and L. Sheng, The Abreu equation with degenerated boundary conditions, *J. Differential Equations* **252** (2012), no. 10, 5235-5259.

[CLS2] B. Chen, A.-M. Li, and L. Sheng, The Kähler metrics of constant scalar curvature on the complex torus, *J. Differential Geom.* **100** (2015), 427-462.

[CLS3] B. Chen, A.-M. Li, and L. Sheng, Extremal metrics on toric surfaces, *Adv. Math.* **340** (2018), 363-405.

[CLS4] B. Chen, A.-M. Li, and L. Sheng, Uniform K-stability for extremal metrics on toric varieties, *J. Differential Equations* **257** (2014), 1487-1500.

[De] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, *Bull. Soc. Math. France* **116** (1988), 315-339.

[Do1] S. K. Donaldson, Scalar curvature and stability of toric varieties, *J. Differential Geom.* **62** (2002), 289-349.

[Do2] S. K. Donaldson, Interior estimates for solutions of Abreu's equation, *Collect. Math.* **56** (2005), 103-142.

[Do3] S. K. Donaldson, Extremal metrics on toric surfaces, *J. Differential Geom.* **79** (2008), 389-432.

[Do4] S. K. Donaldson, Constant scalar curvature metrics on toric surfaces, *Geom. Funct. Anal.* **19** (2009), 83-135.

[Gu] V. Guillemin, Kähler structures on toric varieties, *J. Differential Geom.* **40** (1994), 285-309.

[GS] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping, *Invent. Math.* **67** (1982), 491-513.

[Hu] G. Huang, The Guillemin boundary problem for Monge-Ampère equation in the polygon, *Adv. Math.* **415** (2023), 108885.

[HS] G. Huang and W. Shen, Monge-Ampère equation with Guillemin boundary condition in high dimension, *J. Eur. Math. Soc.*, to appear; arXiv:2406.05471 (2024).

[KLWZ] Y. H. Kim, N. Q. Le, L. Wang, B. Zhou, Singular Abreu equations and linearized Monge–Ampère equations with drifts. *J. Eur. Math. Soc.* (Online first), 2024.

[Le1] N. Q. Le, Global second derivative estimates for the second boundary value problem of the prescribed affine mean curvature and Abreu's equations, *Int. Math. Res. Not. IMRN* **2013**, no. 11, 2421–2438.

[Le2] N. Q. Le, $W^{4,p}$ solution to the second boundary value problem of the prescribed affine mean curvature and Abreu's equations, *J. Differential Equations* **260** (2016), no. 5, 4285–4300.

[Le3] N. Q. Le, Singular Abreu equations and minimizers of convex functionals with a convexity constraint, *Comm. Pure Appl. Math.* **73** (2020), no. 10, 2248–2283.

[Le4] N. Q. Le, On approximating minimizers of convex functionals with a convexity constraint by singular Abreu equations without uniform convexity, *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), no. 1, 356–376.

[Le5] N. Q. Le, On singular Abreu equations in higher dimensions, *J. Anal. Math.* **144** (2021), no. 1, 191–205.

[Le6] N. Q. Le, Twisted Harnack inequality and approximation of variational problems with a convexity constraint by singular Abreu equations, *Adv. Math.* **434** (2023), Paper No. 109325, 31 pp.

[Le7] N. Q. Le, *Analysis of Monge-Ampère Equations*, Graduate Studies in Mathematics, vol. 240, American Mathematical Society, Providence, RI, 2024.

[LZ] N. Q. Le and B. Zhou, Solvability of a class of singular fourth order equations of Monge-Ampère type, *Ann. PDE* **7** (2021), no. 2, Paper No. 13, 32 pp.

[LLS1] A.-M. Li, Z. Lian, and L. Sheng, Some estimates for a generalized Abreu's equation, *Differential Geom. Appl.* **48** (2016), 87-103.

[LLS2] A.-M. Li, Z. Lian, and L. Sheng, Interior regularity for a generalized Abreu equation, *Int. J. Math.* **28** (2017), no. 7, 1750053.

[Ma] T. Mabuchi, K-energy maps integrating Futaki invariants, *Tohoku Math. J.* **38** (1986), 575-593.

[MTW] X.-N. Ma, N. S. Trudinger, and X.-J. Wang, Regularity of potential functions of the optimal transportation problem, *Arch. Ration. Mech. Anal.* **177** (2005), 151-183.

[MTL] H. Mitake, H. V. Tran, and N. Q. Le , *Dynamical and Geometric Aspects of Hamilton-Jacobi and Linearized Monge-Ampère Equations*, Lecture Notes in Mathematics, Vol. 2183, Springer, Cham, 2017.

[Ny] T. Nyberg, Scalar curvature and stability of toric fibrations, *Differential Geom. Appl.* **63** (2019), 186-211.

[Ru] D. Rubin, The Monge-Ampère equation with Guillemin boundary conditions, *Calc. Var. Partial Differential Equations* **54** (2015), 951-968.

[Ti1] G. Tian, Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* **130** (1997), 1-37.

[Ti2] G. Tian, K-stability and Kähler-Einstein metrics, *Comm. Pure Appl. Math.* **68** (2015), no. 7, 1085-1156.

[TW1] N. S. Trudinger and X.-J. Wang, Hessian measures II, *Ann. of Math. (2)* **150** (1999), no. 2, 579-604.

[TW2] N. S. Trudinger and X.-J. Wang, The Bernstein problem for affine maximal hypersurfaces, *Invent. Math.* **140** (2000), no. 2, 399-422.

[TW3] N. S. Trudinger and X.-J. Wang, The affine Plateau problem, *J. Amer. Math. Soc.* **18** (2005), no. 2, 253-289.

[TW4] N. S. Trudinger and X.-J. Wang, The Monge-Ampère equation and its geometric applications, in *Handbook of Geometric Analysis. No. 1*, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, 467-524.

[TW5] N. S. Trudinger and X.-J. Wang, Boundary regularity for the Monge-Ampère and affine maximal surface equations, *Ann. of Math. (2)* **167** (2008), no. 3, 993-1028.

[TW6] N. S. Trudinger and X.-J. Wang, On the second boundary value problem for Monge-Ampère type equations and optimal transportation, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (2009), no. 1, 143-174.

[WZ] L. Wang and B. Zhou, The Guillemin boundary value problem for the Abreu equation, in preparation.

[WZh] X. Wang and X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, *Adv. Math.* **188** (2004), 87-103.

[Yau] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, *Comm. Pure Appl. Math.* **31** (1978), 339-411.

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