A Two-Dimensional Proof of Harnack's Inequality Using the Logarithmic Gradient

The idea is to prove a Dirichlet energy estimate for

$$v = \log u,$$

and then combine a two-dimensional circle-oscillation estimate with the maximum principle. In the notes to Chapter 8, Gilbarg–Trudinger point out that in two dimensions the Holder estimate and Harnack inequality can be obtained by simpler methods, referring to Problem 8.5. That problem asks one to control the Dirichlet integral of $v = \log u$, and then use the weak maximum principle to obtain Harnack.

The following is a cleaned-up version of the argument. To keep the main idea visible, I first treat the core case without lower-order terms:

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } B_1 \subset \mathbb{R}^2,$$

where

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi, \qquad |A(x)\xi| \leq \Lambda|\xi|.$$

Assume $u > 0$. We prove

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u.$$

The core proof

Set

$$v = \log u.$$

The proof has three steps:

$$\boxed{\text{logarithmic energy estimate}} \quad \Longrightarrow \quad \boxed{\text{circle-oscillation estimate}} \quad \Longrightarrow \quad \boxed{\text{maximum principle}}.$$

Step 1: Energy estimate for $\nabla \log u$

Choose $\eta \in C_c^\infty(B_1)$ such that

$$\eta \equiv 1 \quad \text{on } B_{7/8}, \qquad |\nabla\eta| \leq C.$$

Use

$$\varphi = \eta^2 u^{-1}$$

as a test function. Strictly speaking, for weak solutions one may first test with

$$\varphi_\varepsilon = \frac{\eta^2}{u+\varepsilon},$$

and then let $\varepsilon \downarrow 0$.

Since

$$\nabla(\eta^2 u^{-1}) = 2\eta u^{-1}\nabla\eta - \eta^2 u^{-2}\nabla u,$$

the weak formulation

$$\int A\nabla u \cdot \nabla\varphi = 0$$

gives

$$\int \eta^2 A\nabla v \cdot \nabla v = 2\int \eta A\nabla v \cdot \nabla\eta.$$

Using ellipticity and boundedness,

$$\lambda \int \eta^2 |\nabla v|^2 \le 2\Lambda \int \eta |\nabla v|\,|\nabla\eta|.$$

By Young's inequality,

$$\int \eta^2|\nabla v|^2 \le C\int |\nabla\eta|^2.$$

Therefore,

$$\boxed{\int_{B_{7/8}} |\nabla \log u|^2 \le C.}$$

Here $C = C(\lambda, \Lambda)$.

This is the step that uses the "gradient of the logarithm". The special two-dimensional point is that

$$\int_{B_1} |\nabla\eta|^2 \le C$$

is scale-invariant. More generally, if the same estimate is done in B_R, with $|\nabla\eta| \lesssim R^{-1}$, then in two dimensions

$$\int_{B_R} |\nabla\eta|^2 \lesssim R^{-2}|B_R| \lesssim 1.$$

This is precisely where dimension two enters.

Step 2: The two-dimensional circle-oscillation estimate

Define

$$\omega(r) = \operatorname{osc}_{\partial B_r} v = \sup_{\partial B_r} v - \inf_{\partial B_r} v.$$

We first prove a purely two-dimensional lemma.

Lemma 1 (Circle-oscillation estimate)**.** *Suppose $\omega(r)$ is nondecreasing in r. Then for $0 < a < b$,*

$$\boxed{\omega(a)^2 \le \frac{C}{\log(b/a)}\int_{B_b\setminus B_a} |\nabla v|^2.}$$

Proof. For almost every t, on the circle ∂B_t,

$$\omega(t)^2 \le 2\pi \int_0^{2\pi} |v_\theta(t,\theta)|^2 \, d\theta.$$

On the other hand, in polar coordinates,

$$|\nabla v|^2 = v_r^2 + \frac{1}{t^2} v_\theta^2.$$

Hence

$$\int_{B_b \setminus B_a} |\nabla v|^2 \ge \int_a^b \frac{1}{t} \int_0^{2\pi} |v_\theta(t,\theta)|^2 \, d\theta \, dt.$$

Since $\omega(t) \ge \omega(a)$ for $t \ge a$, we get

$$\int_0^{2\pi} |v_\theta(t,\theta)|^2 \, d\theta \ge c\,\omega(a)^2.$$

Therefore,

$$\int_{B_b \setminus B_a} |\nabla v|^2 \ge c\,\omega(a)^2 \int_a^b \frac{dt}{t} = c\,\omega(a)^2 \log \frac{b}{a}.$$

Thus,

$$\omega(a)^2 \le \frac{C}{\log(b/a)} \int_{B_b \setminus B_a} |\nabla v|^2.$$

□

This is the circle-oscillation estimate appearing in Gilbarg–Trudinger, Problem 8.5(a). There, one uses

$$\omega(r) = \operatorname{osc}_{\partial B_r} u, \qquad D(r) = \int_{B_r} |Du|^2,$$

to prove this type of estimate.

Now apply the lemma to

$$v = \log u.$$

Why is $\omega(r)$ nondecreasing? Because u satisfies the maximum principle. Define

$$M(r) = \sup_{\partial B_r} u, \qquad m(r) = \inf_{\partial B_r} u.$$

By the maximum principle, $M(r)$ is nondecreasing in r, while $m(r)$ is nonincreasing in r. Hence

$$\omega(r) = \operatorname{osc}_{\partial B_r} \log u = \log M(r) - \log m(r)$$

is nondecreasing.

Now take

$$a = \frac{3}{4}, \qquad b = \frac{7}{8}.$$

By the circle-oscillation estimate and Step 1,

$$\operatorname{osc}_{\partial B_{3/4}} \log u \le C.$$

Equivalently,

$$\boxed{\frac{\sup_{\partial B_{3/4}} u}{\inf_{\partial B_{3/4}} u} \le e^C.}$$

Step 3: Use the maximum principle to pass to the interior

Since

$$B_{1/2} \subset B_{3/4},$$

the maximum principle gives

$$\sup_{B_{1/2}} u \le \sup_{B_{3/4}} u = \sup_{\partial B_{3/4}} u.$$

Similarly, the minimum principle gives

$$\inf_{B_{1/2}} u \ge \inf_{B_{3/4}} u = \inf_{\partial B_{3/4}} u.$$

Therefore,

$$\frac{\sup_{B_{1/2}} u}{\inf_{B_{1/2}} u} \le \frac{\sup_{\partial B_{3/4}} u}{\inf_{\partial B_{3/4}} u} \le e^C.$$

Thus,

$$\boxed{\sup_{B_{1/2}} u \le C_H \inf_{B_{1/2}} u,}$$

where

$$C_H = C_H(\lambda, \Lambda).$$

After scaling, the general form is:

Theorem 1 (Planar Harnack inequality)**.** *If $u > 0$ solves*

$$\operatorname{div}(A\nabla u) = 0 \qquad \text{in } B_R(x_0) \subset \mathbb{R}^2,$$

with the ellipticity and boundedness assumptions above, then

$$\boxed{\sup_{B_{R/2}(x_0)} u \le C_H \inf_{B_{R/2}(x_0)} u.}$$

The constant C_H depends only on λ and Λ.

Version with first-order terms

Gilbarg–Trudinger, Problem 8.5(b), is formulated for the two-dimensional divergence-structure equation

$$Lu = D_i(a^{ij} D_j u) + b^i D_i u = 0, \qquad i, j = 1, 2,$$

and suggests controlling the Dirichlet integral of $v = \log u$, then using the weak maximum principle and part (a) to obtain Harnack.

If a first-order term b is present, the logarithmic energy estimate gains one additional term:

$$\int \eta^2 |\nabla \log u|^2 \le C \int |\nabla \eta|^2 + C \int \eta^2 |b|^2.$$

Therefore, in B_R,

$$\int_{B_{7R/8}} |\nabla \log u|^2 \le C(1 + R^2 \|b\|_\infty^2).$$

The circle-oscillation estimate and maximum principle then proceed exactly as above, giving

$$\sup_{B_{R/2}} u \le C \inf_{B_{R/2}} u,$$

where

$$C = C(\lambda, \Lambda, R\|b\|_\infty).$$

Conceptual summary

The beautiful two-dimensional proof can be compressed into the chain

$$\boxed{\int |\nabla \log u|^2 \le C \quad \Longrightarrow \quad \operatorname{osc}_{\partial B_r} \log u \le C \quad \Longrightarrow \quad \sup u \le C \inf u.}$$

The first step comes from testing the equation with η^2/u. The second step is a one-dimensional Sobolev/oscillation estimate on circles. The third step is the maximum principle.

The point of the proof is that it avoids Moser iteration. Instead, in two dimensions, Harnack's inequality is reduced to a Dirichlet energy bound for $\log u$.

Reference

D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, 2001. See Chapter 8, Problem 8.5.