INTERIOR $C^{1,\alpha}$ ESTIMATES FOR THE LINEARIZED MONGE–AMPÈRE EQUATION IN TWO DIMENSIONS

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Abstract. We prove an interior $C^{1,\alpha}$ estimate for solutions of the homogeneous linearized Monge–Ampère equation in dimension two under the assumption

$$0 < \lambda \le \det D^2\varphi \le \Lambda < +\infty.$$

No continuity assumption on the Monge–Ampère density is required. Our result is an affine-invariant analogue of the classical Morrey–Nirenberg $C^{1,\alpha}$ estimate in two dimensions. The core of the proof is the partial Legendre transform. After the transform, the first derivatives of the solution are quotients of adjoint solutions for a uniformly elliptic non-divergence form equation. Bauman's Harnack inequality gives the Hölder control of the quotient, while the Jacobian identity of the partial Legendre transform and a Caccioppoli estimate give its local boundedness. As an application, we prove a Liouville theorem for entire solutions with at most linear growth.

1. Introduction

This paper is concerned with the interior $C^{1,\alpha}$ estimate for solutions of the homogeneous linearized Monge–Ampère equation in dimension two. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and φ be a smooth and strictly convex function with

$$0 < \lambda \le \det D^2\varphi \le \Lambda < +\infty \qquad \text{in } \Omega. \tag{1.1}$$

The associated linearized Monge–Ampère operator is

$$L_\varphi u := \Phi^{ij} D_{ij} u, \qquad \Phi := \operatorname{cof}(D^2\varphi) = (\det D^2\varphi)(D^2\varphi)^{-1}.$$

We use the Einstein summation convention throughout the paper. We mainly focus on the homogeneous equation

$$L_\varphi u = 0. \tag{1.2}$$

The difficulty in studying linearized Monge–Ampère equations is that (1.1) usually does not give a uniform ellipticity in fixed Euclidean balls. The natural geometry is instead given by the sections of φ,

$$S_\varphi(x_0, h) := \big\{x \in \Omega : \ \varphi(x) < \varphi(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + h\big\}.$$

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Caffarelli and Gutiérrez [CG97] proved the Harnack inequality and interior Hölder estimate for solutions of the homogeneous linearized Monge–Ampère equation under a $\mathcal{A}_\infty$ condition. In particular, this condition is guaranteed if we have (1.1). Subsequent developments include interior $W^{2,p}$ and second derivative estimates, boundary regularity, global $W^{1,p}$ and $W^{2,p}$ estimates, and applications to semigeostrophic equations. See [GN15, GT06, LN14, LN17, LS13, Le18] and the references therein. Linearized Monge–Ampère operators also arise in affine maximal surface and related fourth-order equations. See, for example, [TW08b, TW08a]. Linearized Monge–Ampère equations with drifts, together with applications to singular Abreu equations, were studied in [KLWZ26]. For the geometry of sections and the interior regularity theory of the Monge–Ampère equation, see [Caf90, Caf91, CG96, GH00, Gut16, Fig17, Le24].

For gradient estimates, the known results require additional information on the Monge–Ampère density. Gutiérrez and Nguyen [GN11] proved interior Hölder estimates for the gradient when $\det D^2\varphi$ is continuous. Tang and Zhang [TZ20] obtained related $C^{1,\alpha}$ estimates under a VMO condition on the density. More recently, Cui [Cui26] established gradient potential estimates and derived criteria for continuity and local $C^{1,\gamma}$ regularity of the gradient. Thus, the known gradient estimates require assumptions on the Monge–Ampère density beyond the two-sided bound (1.1).

The situation in dimension two is different. The classical results of Morrey [Mor38] and Nirenberg [Nir53] give interior $C^{1,\alpha}$ estimates for uniformly elliptic equations in two variables without the continuity assumption on the coefficients. The same first-order estimate does not hold, in general, in higher dimensions [Saf88]. Although L_φ is not uniformly elliptic in Euclidean coordinates, it is affine invariant and the sections of φ provide the natural replacement for Euclidean balls. We therefore ask whether an affine-invariant version of the planar Morrey–Nirenberg estimate holds for the linearized Monge–Ampère equation.

In the positive direction, Savin [Sav10] provided further evidence. In particular, he proved that every entire globally Lipschitz solution of (1.2) in $\mathbb{R}^2$ under the assumption (1.1) is affine. He also observed that this Liouville theorem suggests an interior $C^{1,\alpha}$ estimate in dimension two and indicated that this problem would be addressed in a subsequent work. To the best of our knowledge, however, this estimate has not yet appeared in the literature. In the present paper, we prove such an estimate assuming only (1.1).

After subtracting the tangent plane at the center of a section and applying the standard affine normalization, we assume $\varphi(0) = 0$, $\nabla\varphi(0) = 0$, and write $S_\theta := S_\varphi(0,\theta)$ for $0 < \theta \le 1$. We use the standard notion of a normalized section.

Theorem 1.1 (Interior $C^{1,\alpha}$ estimate). *Let $S_1 \Subset \Omega$ be a normalized section. Assume that φ is smooth and strictly convex in a neighborhood of $\overline{S_1}$ and*

$$0 < \lambda \le \det D^2\varphi \le \Lambda < +\infty \qquad \text{in } S_1.$$

Suppose $u \in C^\infty(S_1) \cap L^\infty(S_1)$ is a solution to (1.2) in S_1. Then for every $\theta \in (0,1)$, there exist $\alpha \in (0,1)$ and $C > 0$, depending only on λ, Λ and θ, such that

$$\|\nabla u\|_{C^\alpha(S_\theta)} \le C \operatorname{osc}_{S_1} u. \tag{1.3}$$

Theorem 1.1 may be viewed as an affine-invariant analogue of the Morrey–Nirenberg estimate. The formulation on normalized sections is essential. The determinant bound (1.1) does not control the Euclidean ellipticity ratio of L_φ, so an estimate with constants depending only on λ and Λ cannot in general be stated on fixed Euclidean balls.

The proof of Theorem 1.1 is based on the partial Legendre transform. This transform is classical in the study of Monge–Ampère equations. To the best of our knowledge, it was first used for the linearized Monge–Ampère equation by the first author in [Wan25], in which a new proof of the Caffarelli–Gutiérrez Hölder estimate in dimension two is given. Here, we apply the transform to the first derivatives of u. Write

$$\varphi^\star(\xi,\eta) := \sup_{x_1}\{x_1\xi - \varphi(x_1,\eta)\}, \qquad (\xi,\eta) = \mathcal{P}(x_1,x_2) := (\varphi_1(x_1,x_2), x_2),$$

and let $\mathcal{Q} = \mathcal{P}^{-1}$. Set

$$a := \varphi^\star_{\xi\xi}, \qquad g := \det D^2\varphi \circ \mathcal{Q}, \qquad \widetilde{u} := u \circ \mathcal{Q}.$$

Then

$$\varphi^\star_{\eta\eta} + g\varphi^\star_{\xi\xi} = 0, \qquad (g\widetilde{u}_\xi)_\xi + \widetilde{u}_{\eta\eta} = 0, \qquad \lambda \le g \le \Lambda. \tag{1.4}$$

Denote

$$L_g := g\partial_{\xi\xi} + \partial_{\eta\eta}, \qquad L_g^* m := \partial_{\xi\xi}(gm) + \partial_{\eta\eta} m.$$

It is easy to see that L_g is uniformly elliptic and L_g^* is its adjoint operator. If

$$p := u_1 \circ \mathcal{Q} = \frac{\widetilde{u}_\xi}{a},$$

then differentiating the two equations in (1.4) gives

$$L_g^* a = 0, \qquad L_g^*(ap) = 0.$$

Hence p is a quotient of two solutions of the same adjoint equation.

Bauman's Harnack inequality for normalized adjoint solutions [Bau84] gives the Hölder control of this quotient once p is locally bounded. It remains to obtain a local bound for p. The Jacobian formula

$$a(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta = \mathrm{d}x$$

and the interior $C^{1,\beta}$ estimate for φ give a lower bound for the mass of a on transformed balls. Since $\widetilde{u}_\xi = ap$, a Caccioppoli estimate for $\widetilde{u}$ then gives a local bound for p. Bauman's Harnack inequality applied to the adjoint quotient yields the Hölder estimate for p. The estimate for u_1 follows by composing with $\mathcal{P}$, and the argument for u_2 is the same after interchanging the two variables.

As an application, we obtain a Liouville theorem in which the global Lipschitz assumption in [Sav10, Theorem 1.1] is replaced by at most linear growth.

Theorem 1.2 (Liouville theorem). *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be smooth and strictly convex, and assume*

$$0 < \lambda \le \det D^2\varphi \le \Lambda < +\infty \qquad \text{in } \mathbb{R}^2.$$

Let $u \in C^\infty(\mathbb{R}^2)$ be a solution to (1.2) in $\mathbb{R}^2$. If

$$|u(x)| \le C_1(1 + |x|) \qquad \text{for all } x \in \mathbb{R}^2,$$

then u is an affine function.

To see the role of Theorem 1.1, normalize the large sections $S_h = S_\varphi(0, h)$ by affine maps with linear parts A_h. The normalized estimate controls $A_h^{-T}\nabla u$ on a fixed section. Instead of estimating the full gradient, we project the gradient difference onto a right singular vector e_h associated with the smallest singular value of A_h. The resulting factor $\|A_h^{-1}\|^{-1}$ compensates for the linear growth of u along the long axis of S_h. Passing to a sequence $h \to \infty$ gives a fixed direction e with $D^2u\, e = 0$ in $\mathbb{R}^2$. Since the dimension is two, the equation then implies $D^2u = 0$.

We work throughout with smooth potentials and smooth solutions, so all changes of variables and differentiations are classical. The estimates are *a priori*. In particular, the constants do not depend on higher derivatives of φ or u.

The rest of the paper is organized as follows. In Section 2, we recall the properties of normalized sections used in the proof. In Section 3, we derive the equations under the partial Legendre transform and the estimate for quotients of adjoint solutions. In Section 4, we prove the local boundedness of the quotient and Theorem 1.1. Theorem 1.2 is proved in Section 5.

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The mathematical problem, the use of the partial Legendre transform, and the overall proof strategy were developed by the authors. All mathematical arguments, results, and cited references were independently checked by the authors, who take full responsibility for the content of the paper.

2. Preliminaries

Throughout the paper, C and c denote positive constants which may change from line to line. Their dependence will be indicated when necessary. For a bounded set E, we write

$$\operatorname{osc}_E u := \sup_E u - \inf_E u, \qquad \|f\|_{C^\alpha(E)} := \|f\|_{L^\infty(E)} + [f]_{C^\alpha(E)}.$$

In Sections 2–4, $S_1 = S_\varphi(0,1)$ is a normalized section as in the introduction and $S_\theta = S_\varphi(0,\theta)$. Unless otherwise stated, we work under the hypotheses of Theorem 1.1.

We first recall the interior $C^{1,\beta}$ estimate and strict convexity of solutions to the Monge–Ampère equation. The strict convexity statement is specific to dimension two.

Lemma 2.1. *Let S_1 be normalized and let φ be convex in S_1 with*

$$\lambda \le \det D^2\varphi \le \Lambda$$

in the Alexandrov sense. For every $\theta \in (0,1)$ there exist $\beta \in (0,1)$ and $C < \infty$, depending only on λ, Λ and θ, such that

$$\|\nabla\varphi\|_{C^\beta(S_\theta)} \le C.$$

Moreover, $\overline{S_\theta} \Subset S_1$ quantitatively, and φ is strictly convex in the interior of S_1.

Proof. The $C^{1,\beta}$ estimate and the quantitative compact inclusion of lower sections follow from Caffarelli's interior theory. See [Caf91, Caf90, Gut16, Fig17, Le24]. In dimension two, interior strict convexity is classical. See [Hei59, Caf90]. ☐

For the partial Legendre transform in the x_1-variable, let

$$\mathcal{P}(x_1,x_2) := (\varphi_1(x_1,x_2), x_2).$$

Since $\varphi_{11} > 0$, the map $\mathcal{P}$ is locally a diffeomorphism. For fixed x_2, the function $x_1 \mapsto \varphi_1(x_1,x_2)$ is strictly increasing. It follows that $\mathcal{P}$ is one-to-one on a section and hence is a diffeomorphism onto its image.

Lemma 2.2. *Let $0 < \theta < \theta_1 < 1$. Denote $Q_h := \mathcal{P}(S_h)$ for $h \le 1$. There exist $d_0 > 0$ and $D_0 < \infty$, depending only on λ, Λ, θ and θ_1, such that*

$$\operatorname{dist}(Q_\theta, \partial Q_{\theta_1}) \ge d_0, \qquad \operatorname{diam} Q_\theta \le D_0. \tag{2.1}$$

The same conclusion holds for the map $(x_1,x_2) \mapsto (x_1, \varphi_2(x_1,x_2))$.

Proof. The diameter bound follows from Lemma 2.1. The first component of $\mathcal{P}$ is φ_1, and the second component is x_2. Both are uniformly bounded on a fixed lower section.

We prove the separation estimate. Since $\overline{S_{\theta_1}} \Subset S_1$, the restriction of $\mathcal{P}$ to $\overline{S_{\theta_1}}$ is a homeomorphism onto its image. Hence

$$\overline{Q_{\theta_1}} = \mathcal{P}(\overline{S_{\theta_1}}), \qquad \partial Q_{\theta_1} = \mathcal{P}(\partial S_{\theta_1}).$$

Suppose by contradiction that the separation estimate fails. Then there are normalized potentials φ_k satisfying the same determinant bounds and points $x^{(k)} \in S_\theta^{(k)}$, $y^{(k)} \in \partial S_{\theta_1}^{(k)}$, such that

$$|\mathcal{P}_k(x^{(k)}) - \mathcal{P}_k(y^{(k)})| \to 0, \qquad \mathcal{P}_k = (\partial_1\varphi_k, x_2). \tag{2.2}$$

Set $\theta_2 = \frac{1+\theta_1}{2}$. By normalized compactness, after passing to a subsequence the potentials converge locally uniformly to a normalized convex Alexandrov solution φ_∞ with the same determinant bounds. The quantitative inclusion of lower sections and Lemma 2.1 give C^1 convergence on a neighborhood of the points under consideration. We may also assume $x^{(k)} \to x_\infty$, $y^{(k)} \to y_\infty$. The convergence of the sections gives

$$\varphi_\infty(x_\infty) \le \theta, \qquad \varphi_\infty(y_\infty) = \theta_1,$$

so $x_\infty \neq y_\infty$. On the other hand, (2.2) and the C^1 convergence imply

$$(x_\infty)_2 = (y_\infty)_2, \qquad (\varphi_\infty)_1(x_\infty) = (\varphi_\infty)_1(y_\infty).$$

The horizontal segment joining x_∞ and y_∞ is contained in $\{\varphi_\infty \le \theta_1\} \Subset S_1$. The restriction of φ_∞ to this segment is a one-dimensional convex C^1 function whose derivative has the same value at the two endpoints. It is therefore affine on the segment, contrary to Lemma 2.1.

The proof for $(x_1, x_2) \mapsto (x_1, \varphi_2)$ is identical. $\square$

Remark 2.3. The separation in Lemma 2.2 can also be obtained from a quantitative modulus of convexity. Recall that, for a differentiable convex function φ, one may define the local modulus of convexity

$$m_\varphi(t; K) := \inf\Big\{\varphi(y) - \varphi(x) - \nabla\varphi(x)\cdot(y-x) : x, y \in K,\ |x-y| \ge t\Big\},$$

where $K \Subset \Omega$. In dimension two, the Aleksandrov–Heinz theorem gives strict convexity for generalized solutions whose Monge–Ampère measure has a positive lower bound. See Heinz [Hei59] and also [TW08a, Remark 3.2] for an elementary proof. In a normalized compact family of sections, this strict convexity is quantitative on compact subsets. More precisely, if K is quantitatively contained in S_1, then for every $t > 0$ there is a number $\omega(t) > 0$, depending only on λ, Λ, t and the relative position of K in S_1, such that

$$m_\varphi(t; K) \ge \omega(t). \tag{2.3}$$

Indeed, otherwise normalized compactness would give a sequence of potentials and pairs of points, staying in a fixed lower section and a fixed positive distance apart, for which the left-hand side tends to zero. After passing to a subsequence, the potentials converge in C^1 on a lower section. The limiting potential would then be affine on a nontrivial segment, contradicting planar strict convexity. For a quantitative formula for the modulus in dimension two, see [Liu21, Lemma 2.5]. Its proof develops the argument in [TW08a, Section 3] and gives a lower bound in terms of a boundary gradient quantity.

Let us indicate how such a modulus gives another proof of the positive separation in Lemma 2.2. Fix $0 < \theta < \theta_1 < 1$ and set $\theta_2 = \frac{1+\theta_1}{2}$. By the compact inclusion of lower sections and Lemma 2.1, there are $C_0 < \infty$, $s_0 > 0$ and $\delta_0 > 0$ such that

$$\{z : \mathrm{dist}(z, \overline{S_{\theta_1}}) \le \delta_0\} \Subset S_{\theta_2}, \qquad \|\nabla\varphi\|_{C^\beta(S_{\theta_2})} \le C_0,$$

and

$$|x - y| \ge s_0 \tag{2.4}$$

whenever $x \in S_\theta$ and $y \in \partial S_{\theta_1}$. We use (2.3) below with $K = \overline{S_{\theta_2}}$. To see the last inequality, note that the segment joining x and y is contained in $\overline{S_{\theta_1}}$, and hence

$$\theta_1 - \theta \le \varphi(y) - \varphi(x) \le C_0|x - y|.$$

Suppose now that $|\mathcal{P}(x) - \mathcal{P}(y)|$ is small for some $x \in S_\theta$ and $y \in \partial S_{\theta_1}$. Then both $|x_2 - y_2|$ and $|\varphi_1(x) - \varphi_1(y)|$ are small. Let $\bar{y} := (y_1, x_2)$. If $|x_2 - y_2|$ is sufficiently small, then $|\bar{y} - y| < \delta_0$. Hence $\bar{y} \in S_{\theta_2}$, and the vertical segment joining y and $\bar{y}$ also lies in S_{θ_2}. The $C^{1,\beta}$ estimate gives

$$|\varphi_1(\bar{y}) - \varphi_1(y)| \le C_0|x_2 - y_2|^\beta. \tag{2.5}$$

It follows that $|\varphi_1(\bar{y}) - \varphi_1(x)|$ is small. On the other hand, (2.4) and the smallness of $|x_2 - y_2|$ imply $|x_1 - y_1| \ge \frac{s_0}{2}$. By convexity, the horizontal segment joining x and $\bar{y}$ lies in S_{θ_2}. Therefore (2.3) gives

$$\varphi(\bar{y}) - \varphi(x) - \nabla\varphi(x) \cdot (\bar{y} - x) \ge \omega(\frac{s_0}{2}) > 0. \tag{2.6}$$

For a differentiable convex function f on an interval,

$$0 \le f(b) - f(a) - f'(a)(b - a) \le |b - a|\,|f'(b) - f'(a)|.$$

Applying this to $f(t) = \varphi(t, x_2)$ and using $|x_1 - y_1| \le \operatorname{diam} S_{\theta_2} \le C$, we obtain from (2.6)

$$|\varphi_1(\bar{y}) - \varphi_1(x)| \ge c > 0,$$

which contradicts (2.5) and the assumed smallness of $|\mathcal{P}(x) - \mathcal{P}(y)|$. This gives another proof of the first estimate in (2.1).

3. Partial Legendre transform and adjoint quotients

We first derive the equation under the partial Legendre transform. Let

$$(\xi, \eta) = \mathcal{P}(x_1, x_2) := (\varphi_1(x_1, x_2), x_2),$$

and write $\mathcal{Q} = \mathcal{P}^{-1}$. The partial Legendre transform of φ in the x_1-variable is

$$\varphi^\star(\xi, \eta) = \sup_{x_1}\{x_1\xi - \varphi(x_1, \eta)\} = x_1\xi - \varphi(x_1, \eta),$$

where $(\xi, \eta) = \mathcal{P}(x_1, \eta)$. We have

$$\varphi^\star_\xi = x_1, \qquad \varphi^\star_\eta = -\varphi_2(\mathcal{Q}(\xi, \eta)).$$

Differentiating once more gives

$$\varphi^\star_{\xi\xi} = \frac{1}{\varphi_{11}}, \qquad \varphi^\star_{\xi\eta} = -\frac{\varphi_{12}}{\varphi_{11}}, \qquad \varphi^\star_{\eta\eta} = -\frac{\det D^2\varphi}{\varphi_{11}}, \tag{3.1}$$

where the derivatives of φ on the right-hand side are evaluated at $\mathcal{Q}(\xi, \eta)$. Set

$$a := \varphi^\star_{\xi\xi} > 0, \qquad c := \varphi^\star_{\xi\eta}, \qquad g := \det D^2\varphi \circ \mathcal{Q}.$$

Then

$$\varphi^\star_{\eta\eta} + g\varphi^\star_{\xi\xi} = 0, \qquad \lambda \le g \le \Lambda. \tag{3.2}$$

We write

$$L_g := g\partial_{\xi\xi} + \partial_{\eta\eta}, \qquad L^*_g m := \partial_{\xi\xi}(gm) + \partial_{\eta\eta} m.$$

Lemma 3.1. *Let u be a solution to $L_\varphi u = 0$ on a section. Denote $p := u_1 \circ \mathcal{Q}$. With the notation above, we have*

$$L^*_g a = 0, \qquad L^*_g(ap) = 0. \tag{3.3}$$

Proof. Let $A := \varphi^\star_\xi = x_1$. Differentiating (3.2) in ξ gives

$$(gA_\xi)_\xi + A_{\eta\eta} = 0.$$

Since $A_\xi = a$, another differentiation in ξ gives

$$\partial_{\xi\xi}(ga) + \partial_{\eta\eta} a = 0.$$

This is the first equation in (3.3).

Since Φ is divergence free and sections are simply connected, there is a function v, unique up to an additive constant, such that

$$\nabla v = J\Phi\nabla u, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$v_1 = \varphi_{12}u_1 - \varphi_{11}u_2, \qquad v_2 = \varphi_{22}u_1 - \varphi_{12}u_2. \tag{3.4}$$

Define

$$\widetilde{u} := u \circ \mathcal{Q}, \qquad \widetilde{v} := v \circ \mathcal{Q}.$$

Since

$$x_{1,\xi} = \frac{1}{\varphi_{11}}, \qquad x_{1,\eta} = -\frac{\varphi_{12}}{\varphi_{11}},$$

we have

$$\widetilde{u}_\xi = \frac{u_1}{\varphi_{11}}, \qquad \widetilde{u}_\eta = u_2 - \frac{\varphi_{12}}{\varphi_{11}}u_1. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\widetilde{v}_\xi = -\widetilde{u}_\eta, \qquad \widetilde{v}_\eta = g\widetilde{u}_\xi.$$

Hence

$$(g\widetilde{u}_\xi)_\xi + \widetilde{u}_{\eta\eta} = 0. \tag{3.6}$$

By (3.1) and (3.5), $\widetilde{u}_\xi = ap$. Differentiating (3.6) in ξ gives

$$\partial_{\xi\xi}(gap) + \partial_{\eta\eta}(ap) = 0,$$

which is the second equation. □

Remark 3.2. There is also a divergence form equation for p. If $F = (\varphi^\star_\xi, -\varphi^\star_\eta)$, then

$$DF = \begin{pmatrix} a & c \\ -c & ga \end{pmatrix}$$

and (3.3) gives

$$\operatorname{div}(\operatorname{cof} DF\, \nabla p) = 0, \tag{3.7}$$

where

$$\operatorname{cof} DF = \begin{pmatrix} ga & c \\ -c & a \end{pmatrix}$$

is the cofactor of DF. Indeed, using $c_\eta = -(ga)_\xi$ and $c_\xi = a_\eta$, the left-hand side equals

$$ga\, p_{\xi\xi} + a\, p_{\eta\eta} + 2(ga)_\xi p_\xi + 2a_\eta p_\eta,$$

which is the equation obtained from $L^*_g(ap) - pL^*_g a = 0$. However, equation (3.7) is not uniformly elliptic, since the symmetric part of cof DF is

$$\begin{pmatrix} ga & 0 \\ 0 & a \end{pmatrix}$$

and a has no uniform upper or lower bound. Dividing by a gives

$$gp_{\xi\xi} + p_{\eta\eta} + 2\frac{(ga)_\xi}{a}p_\xi + 2\frac{a_\eta}{a}p_\eta = 0.$$

Here $\frac{(ga)_\xi}{a} = g_\xi + g\frac{a_\xi}{a}$. So the lower order terms involve derivatives of g. No estimate for these terms follows from (1.1). We therefore keep the adjoint system (3.3), which avoids differentiating g.

We recall Bauman's Harnack inequality that will be used below. Let $D \subset \mathbb{R}^n$ be a smooth bounded domain, and let $\mathcal{L} = b^{ij}D_{ij}$ with smooth symmetric coefficients satisfying

$$\lambda_0|\zeta|^2 \le b^{ij}(z)\zeta_i\zeta_j \le \Lambda_0|\zeta|^2 \qquad \text{for all } z \in D,\ \zeta \in \mathbb{R}^n.$$

Fix $x_0 \in D$, and let $G_D(x_0,\cdot)$ denote the positive Green function of $\mathcal{L}$ in D with pole at x_0. In the second variable,

$$\mathcal{L}^* G_D(x_0,\cdot) = 0 \qquad \text{in } D \setminus \{x_0\},$$

where

$$\mathcal{L}^* v := D_{ij}(b^{ij}v).$$

Following Bauman [Bau84], a function q on an open set $U \Subset D \setminus \{x_0\}$ is called a *normalized adjoint solution* relative to $G_D(x_0, \cdot)$ if

$$\mathcal{L}^*\big(qG_D(x_0, \cdot)\big) = 0 \qquad \text{in } U.$$

Since $G_D(x_0, \cdot)$ itself solves the adjoint equation away from its pole, the normalized adjoint solutions form a vector space containing the constants. In particular, if q is a normalized adjoint solution and $c \in \mathbb{R}$, then $q - c$ is again a normalized adjoint solution on the same set. This invariance allows one to pass directly from Bauman's Harnack inequality to a quantitative oscillation estimate.

Theorem 3.3. *Let $B_{2r}(z_0) \Subset D \setminus \{x_0\}$ and let q be a smooth normalized adjoint solution in $B_{2r}(z_0)$. If $q \geq 0$ in $B_{2r}(z_0)$, then*

$$\sup_{B_r(z_0)} q \leq H \inf_{B_r(z_0)} q, \tag{3.8}$$

where $H \geq 1$ depends only on n, λ_0, and Λ_0.

For a normalized adjoint solution of arbitrary sign, there exist $\gamma \in (0, 1)$ and $C < \infty$, depending only on the same data, such that

$$r^\gamma [q]_{C^\gamma(B_r(z_0))} \leq C \operatorname{osc}_{B_{2r}(z_0)} q. \tag{3.9}$$

The constants depend only on n, λ_0 and Λ_0.

Proof. Bauman's Harnack principle [Bau84, Theorem 4.4] is stated with $B_{4r}(z_0)$ in place of $B_{2r}(z_0)$. The form (3.8) follows from that result by a finite Harnack chain inside $B_{2r}(z_0)$. We prove the Hölder estimate from this form of the Harnack inequality. See also [Bau84, Theorem 4.5].

Let $B_{2s}(z) \Subset B_{2r}(z_0)$ and introduce the notation

$$m := \inf_{B_{2s}(z)} q, \qquad M := \sup_{B_{2s}(z)} q,$$

and

$$m_1 := \inf_{B_s(z)} q, \qquad M_1 := \sup_{B_s(z)} q.$$

Both $q - m$ and $M - q$ are normalized adjoint solutions in $B_{2s}(z)$. They are nonnegative there, so (3.8) gives

$$M_1 - m \leq H(m_1 - m), \tag{3.10}$$

$$M - m_1 \leq H(M - M_1). \tag{3.11}$$

Set

$$\omega := M - m = \operatorname{osc}_{B_{2s}(z)} q, \qquad \omega_1 := M_1 - m_1 = \operatorname{osc}_{B_s(z)} q.$$

Since

$$M_1 - m = (m_1 - m) + \omega_1,$$

inequality (3.10) implies

$$\omega_1 \leq (H - 1)(m_1 - m).$$

Similarly, (3.11) implies

$$\omega_1 \le (H-1)(M-M_1).$$

If $H=1$, then $\omega_1 = 0$. If $H>1$, the preceding inequalities give

$$m_1 - m \ge \frac{\omega_1}{H-1}, \qquad M - M_1 \ge \frac{\omega_1}{H-1}.$$

Using

$$\omega = (m_1 - m) + \omega_1 + (M - M_1),$$

we conclude in both cases that

$$\operatorname{osc}_{B_s(z)} q \le \vartheta \operatorname{osc}_{B_{2s}(z)} q, \tag{3.12}$$

where $\vartheta := \frac{H-1}{H+1} \in [0,1)$.

Fix $x \in B_r(z_0)$. Since $\overline{B_r(x)} \subset B_{2r}(z_0)$, we may apply (3.12) successively to the concentric balls $B_r(x),\ B_{r/2}(x),\ B_{r/4}(x), \cdots$. Thus, for every integer $k \ge 0$,

$$\operatorname{osc}_{B_{2^{-k}r}(x)} q \le \vartheta^k \operatorname{osc}_{B_r(x)} q \le \vartheta^k \operatorname{osc}_{B_{2r}(z_0)} q. \tag{3.13}$$

Choose $\gamma \in (0,1)$, depending only on H, such that $\vartheta \le 2^{-\gamma}$. For instance, if $\vartheta > 0$, one may take

$$\gamma = \min\left\{\frac{1}{2}, -\frac{\log \vartheta}{\log 2}\right\},$$

while if $\vartheta = 0$, one may take $\gamma = \frac{1}{2}$.

Given $0 < \rho \le r$, choose $k \ge 0$ such that

$$2^{-(k+1)} r < \rho \le 2^{-k} r.$$

It follows from (3.13) that

$$\operatorname{osc}_{B_\rho(x)} q \le C \left(\frac{\rho}{r}\right)^\gamma \operatorname{osc}_{B_{2r}(z_0)} q. \tag{3.14}$$

Let $x, y \in B_r(z_0)$. If $0 < |x-y| < r/2$, then $y \in B_{2|x-y|}(x)$, and (3.14) yields

$$|q(x) - q(y)| \le C \left(\frac{|x-y|}{r}\right)^\gamma \operatorname{osc}_{B_{2r}(z_0)} q.$$

If $|x-y| \ge r/2$, the same estimate follows from

$$|q(x) - q(y)| \le \operatorname{osc}_{B_{2r}(z_0)} q \le 2^\gamma \left(\frac{|x-y|}{r}\right)^\gamma \operatorname{osc}_{B_{2r}(z_0)} q.$$

Taking the supremum over $x, y \in B_r(z_0)$ proves (3.9). $\square$

We shall use the following consequence of Theorem 3.3. The numerator and denominator are normalized by the same Green function, which cancels in the quotient.

Corollary 3.4. *Let $g \in C^\infty(B_{8r}(z_0))$ satisfy*

$$\lambda \le g \le \Lambda \qquad \text{in } B_{8r}(z_0).$$

Let $a > 0$ and w be smooth functions satisfying

$$L_g^* a = 0, \qquad L_g^* w = 0 \qquad \text{in } B_{8r}(z_0).$$

Suppose that $|w| \le Ma$ in $B_{2r}(z_0)$ for some $M \ge 0$. Then there exist $\gamma \in (0,1)$ and $C < \infty$, depending only on λ and Λ, such that

$$\left\| \frac{w}{a} \right\|_{L^\infty(B_r(z_0))} + r^\gamma \left[\frac{w}{a} \right]_{C^\gamma(B_r(z_0))} \le CM. \tag{3.15}$$

In particular, if $w = ap$, then

$$\|p\|_{L^\infty(B_r(z_0))} + r^\gamma [p]_{C^\gamma(B_r(z_0))} \le C \|p\|_{L^\infty(B_{2r}(z_0))}.$$

No modulus of continuity or higher norm of g enters the estimate.

Proof. By translation and scaling, it is enough to consider $z_0 = 0$ and $r = 1$. The case $M = 0$ is immediate.

Choose $x_0 \in B_8 \setminus \overline{B_6}$, and let $G = G_{B_8}(x_0, \cdot)$ be the Green function used in the definition of normalized adjoint solutions. Set

$$q_0 := \frac{a}{G} \qquad \text{in } B_4,$$

and

$$q_\pm := \frac{Ma \pm w}{G} \qquad \text{in } B_2.$$

Then q_0 is a positive normalized adjoint solution in B_4, while q_+ and q_- are nonnegative normalized adjoint solutions in B_2. Moreover,

$$q_+ + q_- = 2Mq_0, \qquad \frac{w}{a} = \frac{q_+ - q_-}{2q_0}. \tag{3.16}$$

Applying the Harnack estimate in Theorem 3.3 to q_0 on $B_2 \subset B_4$, we obtain

$$C^{-1} q_0(0) \le q_0 \le C q_0(0) \qquad \text{in } B_2. \tag{3.17}$$

In particular,

$$\operatorname{osc}_{B_2} q_0 \le C q_0(0).$$

Since $q_\pm \ge 0$ and $q_+ + q_- = 2Mq_0$,

$$0 \le q_\pm \le CMq_0(0) \qquad \text{in } B_2,$$

and hence

$$\operatorname{osc}_{B_2} q_\pm \le CMq_0(0).$$

The Hölder estimate in Theorem 3.3, applied on $B_1 \subset B_2$, therefore gives

$$[q_0]_{C^\gamma(B_1)} \le C q_0(0), \qquad [q_\pm]_{C^\gamma(B_1)} \le CMq_0(0).$$

Let $N := q_+ - q_-$. Then

$$\|N\|_{L^\infty(B_1)} + [N]_{C^\gamma(B_1)} \le CMq_0(0),$$

whereas (3.17) implies

$$\inf_{B_1} q_0 \geq C^{-1} q_0(0).$$

The elementary quotient estimate

$$\left[\frac{N}{q_0}\right]_{C^\gamma(B_1)} \leq \frac{[N]_{C^\gamma(B_1)}}{\inf_{B_1} q_0} + \frac{\|N\|_{L^\infty(B_1)} \cdot [q_0]_{C^\gamma(B_1)}}{(\inf_{B_1} q_0)^2}$$

now yields

$$\left[\frac{N}{q_0}\right]_{C^\gamma(B_1)} \leq CM.$$

Using (3.16), we conclude that

$$\left[\frac{w}{a}\right]_{C^\gamma(B_1)} \leq CM.$$

The L^∞ estimate follows directly from $|w| \leq Ma$. Rescaling proves (3.15). The final assertion follows by taking $w = ap$ and $M = \|p\|_{L^\infty(B_{2r}(z_0))}$. □

4. Local boundedness of the quotient and proof of Theorem 1.1

Let $Q = \mathcal{P}(S_1)$. If $E \subset Q$ is measurable, then

$$\mathrm{d}\xi \mathrm{d}\eta = \varphi_{11}(x)\, \mathrm{d}x, \qquad a(\xi,\eta) = \frac{1}{\varphi_{11}(\mathcal{Q}(\xi,\eta))}.$$

Hence

$$\int_E a(\xi,\eta)\, \mathrm{d}\xi \mathrm{d}\eta = |\mathcal{Q}(E)|. \tag{4.1}$$

This identity will be used below.

Lemma 4.1. *Let S_1 be normalized, set $Q_h := \mathcal{P}(S_h)$, and fix $0 < \theta_1 < 1$. Then there exist $\beta \in (0,1)$ and $c_0, r_0 > 0$, depending only on λ, Λ and θ_1, such that for every $z \in Q_{\theta_1}$ and every $0 < r \leq r_0$ satisfying $B_r(z) \subset Q_{\theta_1}$,*

$$\int_{B_r(z)} a\, \mathrm{d}\xi \mathrm{d}\eta \geq c_0 r^{2/\beta}.$$

Proof. Set $\theta_2 = \frac{1+\theta_1}{2}$. The quantitative compact inclusion of sections gives

$$\overline{S_{\theta_1}} \Subset S_{\theta_2} \Subset S_1$$

with uniform separation. By Lemma 2.1, there are $\beta \in (0,1)$ and $C_0 < \infty$ such that

$$|\mathcal{P}(x) - \mathcal{P}(y)| \leq C_0 |x-y|^\beta \qquad \text{for } x, y \in S_{\theta_2}. \tag{4.2}$$

Choose $r_0 > 0$ so small that

$$B_{(\frac{r}{2C_0})^{1/\beta}}(x) \subset S_{\theta_2}$$

for every $x \in \overline{S_{\theta_1}}$ and $0 < r \le r_0$. Let $z = \mathcal{P}(x) \in Q_{\theta_1}$. If $|y - x| \le (\frac{r}{2C_0})^{1/\beta}$, then (4.2) gives $|\mathcal{P}(y) - \mathcal{P}(x)| < r$. Since $\mathcal{P}$ is one-to-one,

$$B_{(\frac{r}{2C_0})^{1/\beta}}(x) \subset \mathcal{Q}(B_r(z)).$$

The mass identity (4.1) therefore yields

$$\int_{B_r(z)} a \,\mathrm{d}\xi\mathrm{d}\eta = |\mathcal{Q}(B_r(z))| \ge c_0 r^{2/\beta}.$$

□

Lemma 4.2. *Let w satisfy*

$$(g w_\xi)_\xi + w_{\eta\eta} = 0$$

in $B_{2r}(z_0)$, where $\lambda \le g \le \Lambda$. Then

$$\int_{B_r(z_0)} |w_\xi| \,\mathrm{d}\xi\mathrm{d}\eta \le Cr \operatorname{osc}_{B_{2r}(z_0)} w,$$

where C depends only on λ and Λ.

Proof. By the Caccioppoli inequality,

$$\int_{B_r(z_0)} |\nabla w|^2 \,\mathrm{d}\xi\mathrm{d}\eta \le \frac{C}{r^2} \int_{B_{2r}(z_0)} |w - w_{B_{2r}}|^2 \,\mathrm{d}\xi\mathrm{d}\eta \le C(\operatorname{osc}_{B_{2r}(z_0)} w)^2.$$

By the Cauchy–Schwarz inequality,

$$\int_{B_r(z_0)} |w_\xi| \,\mathrm{d}\xi\mathrm{d}\eta \le |B_r|^{1/2} \left(\int_{B_r(z_0)} |w_\xi|^2 \,\mathrm{d}\xi\mathrm{d}\eta \right)^{1/2} \le Cr \operatorname{osc}_{B_{2r}(z_0)} w.$$

□

Proposition 4.3. *Let S_1 be normalized and let $0 < \theta < 1$. Then*

$$\|p\|_{L^\infty(\mathcal{P}(S_\theta))} \le C \operatorname{osc}_{S_1} u,$$

where C depends only on λ, Λ and θ.

Proof. Set

$$\theta_* := \frac{1+\theta}{2}, \qquad Q_* := \mathcal{P}(S_{\theta_*}).$$

Let β, c_0, r_0 be the constants in Lemma 4.1 for θ_*. For $z \in Q_*$, set

$$d(z) := \operatorname{dist}(z, \partial Q_*), \qquad D(z) := \min\{d(z), r_0\}.$$

Choose an integer N such that

$$N > \frac{2}{\beta} - 1.$$

Note that $\overline{Q_*} \Subset Q$ and p is smooth. While $D = 0$ on ∂Q_*, the function $|p| D^N$ attains its maximum on $\overline{Q_*}$. Let $y \in Q_*$ be a maximum point. The conclusion is immediate if this maximum is zero, so suppose $|p(y)| \cdot D(y)^N > 0$.

For $z \in B_{\frac{D(y)}{2}}(y)$, the 1-Lipschitz property of the distance function gives $D(z) \geq \frac{D(y)}{2}$. By the maximality,

$$|p(z)| \leq 2^N |p(y)| \qquad \text{in } B_{\frac{D(y)}{2}}(y).$$

Define

$$q(z) := \frac{p(z)}{2^N |p(y)|}, \qquad R_y := \frac{D(y)}{16}.$$

Then $|q| \leq 1$ in $B_{8R_y}(y)$ and $|q(y)| = 2^{-N}$. If $w_q := aq$, then

$$L_g^* a = 0, \qquad L_g^* w_q = 0 \qquad \text{in } B_{8R_y}(y).$$

Corollary 3.4, applied with $M = 1$, gives

$$R_y^\gamma \cdot [q]_{C^\gamma(B_{R_y}(y))} \leq C.$$

Choose $\tau \in (0, \frac{1}{16})$, depending only on N, λ, Λ, so that $C(16\tau)^\gamma \leq 2^{-N-1}$. If $|z - y| \leq \tau D(y)$, then

$$|q(z) - q(y)| \leq C \frac{|z-y|^\gamma}{R_y^\gamma} \leq C(16\tau)^\gamma \leq 2^{-N-1}.$$

Since $|q(y)| = 2^{-N}$, we obtain

$$|p(z)| \geq \frac{1}{2} |p(y)| \qquad \text{in } B_r(y), \qquad r := \tau D(y).$$

The ball $B_r(y)$ is contained in Q_* and $r \leq r_0$, so Lemma 4.1 and the identity $\widetilde{u}_\xi = ap$ imply

$$\int_{B_r(y)} |\widetilde{u}_\xi| \, \mathrm{d}\xi \mathrm{d}\eta \geq c|p(y)| r^{2/\beta}.$$

Since $2r = 2\tau D(y) < D(y)$, the ball $B_{2r}(y)$ is compactly contained in Q_*. Lemma 4.2 therefore gives

$$\int_{B_r(y)} |\widetilde{u}_\xi| \, \mathrm{d}\xi \mathrm{d}\eta \leq Cr \operatorname{osc}_{B_{2r}(y)} \widetilde{u} \leq Cr \operatorname{osc}_{S_1} u.$$

Consequently,

$$|p(y)| \leq C D(y)^{1-2/\beta} \operatorname{osc}_{S_1} u. \tag{4.3}$$

By Lemma 2.2, $\delta := \inf_{\mathcal{P}(S_\theta)} D > 0$ with a lower bound depending only on λ, Λ and θ. For $z \in \mathcal{P}(S_\theta)$, maximality of y and (4.3) give

$$|p(z)| \leq \delta^{-N} |p(y)| D(y)^N \leq C\delta^{-N} D(y)^{N+1-2/\beta} \operatorname{osc}_{S_1} u.$$

Because $N + 1 - \frac{2}{\beta} > 0$ and $D(y) \leq r_0$, the right-hand side is bounded by $C \operatorname{osc}_{S_1} u$. This proves the proposition. ☐

Proposition 4.4. *Let S_1 be normalized and let $0 < \theta < 1$. Then*

$$\|p\|_{C^\gamma(\mathcal{P}(S_\theta))} \leq C \operatorname{osc}_{S_1} u,$$

where $\gamma \in (0,1)$ depends only on λ and Λ, and C depends only on λ, Λ and θ.

Proof. Set $\theta_1 = \frac{1+\theta}{2}$ and $Q_h := \mathcal{P}(S_h)$. Lemma 2.2 gives a quantitative separation between Q_θ and ∂Q_{θ_1}. Proposition 4.3, applied at the level θ_1, gives

$$\|p\|_{L^\infty(Q_{\theta_1})} \le C \operatorname{osc}_{S_1} u.$$

By Lemma 2.2, there exists $r_* > 0$, depending only on λ, Λ and θ, such that

$$B_{8r_*}(z) \subset Q_{\theta_1} \qquad \text{for every } z \in Q_\theta.$$

For each $z \in Q_\theta$, apply Corollary 3.4 with $w = ap$ in $B_{8r_*}(z)$. Since

$$\|p\|_{L^\infty(B_{2r_*}(z))} \le \|p\|_{L^\infty(Q_{\theta_1})} \le C \operatorname{osc}_{S_1} u,$$

we obtain

$$\|p\|_{L^\infty(B_{r_*}(z))} + r_*^\gamma [p]_{C^\gamma(B_{r_*}(z))} \le C \operatorname{osc}_{S_1} u.$$

We now pass from the local estimate to Q_θ. If $z_1, z_2 \in Q_\theta$ and $|z_1 - z_2| < r_*$, then the preceding estimate applied at z_1 gives

$$|p(z_1) - p(z_2)| \le C \operatorname{osc}_{S_1} u \, |z_1 - z_2|^\gamma.$$

If $|z_1 - z_2| \ge r_*$, then

$$|p(z_1) - p(z_2)| \le 2\|p\|_{L^\infty(Q_{\theta_1})} \le C \operatorname{osc}_{S_1} u \le C r_*^{-\gamma} \operatorname{osc}_{S_1} u \, |z_1 - z_2|^\gamma.$$

Since r_* is quantitatively controlled, this gives

$$\|p\|_{C^\gamma(Q_\theta)} \le C \operatorname{osc}_{S_1} u.$$

□

Proof of Theorem 1.1. We first estimate u_1. Proposition 4.4 gives

$$\|p\|_{C^\gamma(\mathcal{P}(S_\theta))} \le C \operatorname{osc}_{S_1} u.$$

By Lemma 2.1, the map $\mathcal{P}(x) = (\varphi_1(x), x_2)$ is C^β on S_θ, with

$$|\mathcal{P}(x) - \mathcal{P}(y)| \le C|x-y|^\beta \qquad \text{for } x, y \in S_\theta.$$

Since $u_1 = p \circ \mathcal{P}$, we obtain

$$\|u_1\|_{C^{\beta\gamma}(S_\theta)} \le C \operatorname{osc}_{S_1} u. \tag{4.4}$$

The constants depend only on λ, Λ and θ.

We apply the same argument in the second variable. Write $(\xi, \eta) = (x_1, \varphi_2(x_1, x_2))$, let $\widehat{\mathcal{Q}}$ be the inverse map, and set

$$\widehat{a} = (\varphi_{22} \circ \widehat{\mathcal{Q}})^{-1}, \qquad \widehat{p} = u_2 \circ \widehat{\mathcal{Q}}, \qquad \widehat{g} = \det D^2\varphi \circ \widehat{\mathcal{Q}}.$$

The transformed operator is $\partial_{\xi\xi} + \widehat{g}\partial_{\eta\eta}$, and $\widehat{a}$ and $\widehat{a}\widehat{p}$ solve its adjoint equation. The mass identity and all preceding estimates are unchanged after interchanging the two coordinates. Hence

$$\|u_2\|_{C^{\beta\gamma}(S_\theta)} \le C \operatorname{osc}_{S_1} u. \tag{4.5}$$

Combining (4.4) and (4.5) proves (1.3) with $\alpha = \beta\gamma$. □

5. Proof of Theorem 1.2

Proof of Theorem 1.2. After subtracting the tangent plane of φ at the origin, we may assume

$$\varphi(0) = 0, \qquad \nabla\varphi(0) = 0,$$

and write

$$S_h := S_\varphi(0,h) = \{x \in \mathbb{R}^2 : \varphi(x) < h\}.$$

Every S_h is bounded. Indeed, suppose that $\overline{S_h}$ were unbounded. Choose $x_k \in \overline{S_h}$ with $|x_k| \to \infty$ and, after passing to a subsequence, assume $\frac{x_k}{|x_k|} \to v \in \mathbb{S}^1$. Since $0 \in \overline{S_h}$ and $\overline{S_h}$ is convex, for every fixed $t > 0$ one has $\frac{t}{|x_k|}x_k \in \overline{S_h}$ for all sufficiently large k. Passing to the limit gives $tv \in \overline{S_h}$. Thus $t \mapsto \varphi(tv)$ is convex, nonnegative, has right derivative zero at $t = 0$, and is bounded above by h on $[0,\infty)$. Its derivative must therefore vanish identically, contradicting strict convexity. The sections are increasing, and they exhaust $\mathbb{R}^2$ because φ is finite everywhere.

For each $h > 0$, let $\mathcal{A}_h x = A_h x + c_h$ be an orientation-preserving affine normalization of S_h. By the standard normalization and volume estimates for Monge–Ampère sections, see for example [GH00, Gut16, Le24], we may arrange that, with $\widetilde{x}_h := \mathcal{A}_h(0)$,

$$B_{c_0}(\widetilde{x}_h) \subset \mathcal{A}_h S_h \subset B_{C_0}(\widetilde{x}_h), \qquad \det A_h = h^{-1}, \tag{5.1}$$

where $0 < c_0 < C_0 < \infty$ depend only on λ and Λ. Starting from a standard normalization, the determinant can be fixed exactly by multiplying the linear part by a scalar bounded above and below by structural constants, since $|S_h| \simeq h$ in dimension two. We claim that

$$\|A_h\| \to 0 \qquad \text{as } h \to \infty. \tag{5.2}$$

Fix $R > 0$. Since S_h exhausts $\mathbb{R}^2$, one has $B_R \subset S_h$ for all sufficiently large h. For every unit vector v, both 0 and Rv then belong to S_h, and hence

$$R|A_h v| = |\mathcal{A}_h(Rv) - \mathcal{A}_h(0)| \le \operatorname{diam}(\mathcal{A}_h S_h) \le 2C_0.$$

Thus $\|A_h\| \le \frac{2C_0}{R}$ eventually. Since R is arbitrary, (5.2) follows.

Define

$$\widetilde{\varphi}_h(y) := \frac{1}{h}\varphi(\mathcal{A}_h^{-1}y), \qquad \widetilde{u}_h(y) := u(\mathcal{A}_h^{-1}y).$$

Then

$$\widetilde{\varphi}_h(\widetilde{x}_h) = 0, \qquad \nabla\widetilde{\varphi}_h(\widetilde{x}_h) = 0,$$

and

$$S_{\widetilde{\varphi}_h}(\widetilde{x}_h, 1) = \mathcal{A}_h S_h, \qquad S_{\widetilde{\varphi}_h}(\widetilde{x}_h, 1/2) = \mathcal{A}_h S_{h/2}.$$

Moreover, because $\det A_h = h^{-1}$,

$$\det D_y^2 \widetilde{\varphi}_h(y) = \det D_x^2 \varphi(\mathcal{A}_h^{-1}y),$$

and affine covariance gives

$$L_{\widetilde{\varphi}_h} \widetilde{u}_h = 0 \qquad \text{in } \mathcal{A}_h S_h.$$

After translating the y-variables by $-\widetilde{x}_h$, the section in (5.1) is normalized. Fix the exponent $\alpha_0 \in (0,1)$ supplied by Theorem 1.1 at the relative scale $1/2$. Applying that theorem to $\widetilde{u}_h$ yields

$$\left|A_h^{-T}\big(\nabla u(x) - \nabla u(x')\big)\right| \le C \operatorname{osc}_{S_h} u \cdot |A_h(x - x')|^{\alpha_0} \tag{5.3}$$

for every $x, x' \in S_{h/2}$, with C independent of h. Here A_h^{-T} denotes the transpose of A_h^{-1}.

For each h, choose a unit right singular vector e_h of A_h associated with its smallest singular value. Thus

$$|A_h e_h| = \sigma_{\min}(A_h) = \|A_h^{-1}\|^{-1}. \tag{5.4}$$

Using

$$\big(\nabla u(x) - \nabla u(x')\big) \cdot e_h = \big(A_h^{-T}\big(\nabla u(x) - \nabla u(x')\big)\big) \cdot (A_h e_h),$$

together with (5.3) and (5.4), we obtain

$$\left|\big(\nabla u(x) - \nabla u(x')\big) \cdot e_h\right| \le C \frac{\operatorname{osc}_{S_h} u}{\|A_h^{-1}\|} \cdot \|A_h\|^{\alpha_0} \cdot |x - x'|^{\alpha_0}. \tag{5.5}$$

The translation part of $\mathcal{A}_h$ cancels when distances are taken, and (5.1) implies

$$\operatorname{diam} S_h \le C\|A_h^{-1}\|.$$

Since $0 \in S_h$, the linear growth assumption therefore yields

$$\operatorname{osc}_{S_h} u \le C\big(1 + \|A_h^{-1}\|\big).$$

By (5.2), we have $\|A_h^{-1}\| \to \infty$, and

$$\frac{\operatorname{osc}_{S_h} u}{\|A_h^{-1}\|} \le C \tag{5.6}$$

for all sufficiently large h.

Choose a sequence $h_k \to \infty$. After passing to a subsequence, compactness of the unit circle gives $e_{h_k} \to e_\infty \in \mathbb{S}^1$. Fix arbitrary $x, x' \in \mathbb{R}^2$. For all sufficiently large k, one has $x, x' \in S_{\frac{h_k}{2}}$. Combining (5.5), (5.6), and (5.2), and then letting $k \to \infty$, gives

$$\big(\nabla u(x) - \nabla u(x')\big) \cdot e_\infty = 0.$$

Hence $e_\infty \cdot \nabla u$ is constant in $\mathbb{R}^2$. Therefore

$$D^2 u \, e_\infty = 0 \qquad \text{in } \mathbb{R}^2. \tag{5.7}$$

Let $e_\infty^\perp$ be a unit vector orthogonal to e_∞. Since D^2u is a symmetric 2×2 matrix, (5.7) implies that, pointwise,

$$D^2 u = \mu \, e_\infty^\perp \otimes e_\infty^\perp$$

for a scalar function μ. Substitution into the equation yields

$$0 = L_\varphi u = \operatorname{tr}(\Phi D^2 u) = \mu \, (e_\infty^\perp)^T \Phi e_\infty^\perp.$$

The cofactor matrix Φ is positive definite, so $(e_\infty^\perp)^T \Phi e_\infty^\perp > 0$. Consequently $\mu \equiv 0$, hence $D^2 u \equiv 0$ and u is affine. □

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