LIOUVILLE-TYPE THEOREMS FOR COUPLED-DRIFT MONGE–AMPÈRE EQUATIONS

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ABSTRACT. We study entire solutions and periodic correctors for the coupled-drift Monge–Ampère equation

$$\det D^2u = \exp\{-a\cdot Du + b\cdot x + V(x) - c_0\},\quad D^2u > 0.$$

For $V\equiv 0$, we obtain a sharp classification of the whole-space solvability regimes: all entire smooth strictly convex solutions are quadratic when $a=b=0$; no such solution exists when $a\neq 0$ and $a\cdot b\leq 0$; and non-quadratic entire solutions exist when $a=0$ and $b\neq 0$, or when $a\cdot b>0$. The main new ingredient is a scalar maximum-principle argument valid in every dimension $n\geq 2$, which proves that the null case $a\neq 0$, $a\cdot b=0$ admits no entire smooth strictly convex solution.

For periodic V, we establish the existence and uniqueness for the drifted cell problem

$$\det(A + D^2\psi) = \exp\{-a\cdot D\psi + V - c_A\},\quad A + D^2\psi > 0\quad \text{on } \mathbb{T}^n.$$

We also prove that any asymptotically quadratic entire solution must satisfy $b = Aa$. If its remainder is bounded, then the solution is the corresponding quadratic-periodic corrector up to an additive constant.

1. INTRODUCTION

This paper studies global convex solutions of Monge–Ampère equations whose exponential density contains two linear first-order drifts. The model equation is

$$\det D^2u = \exp\{-a\cdot Du + b\cdot x + V(x) - c_0\},\quad D^2u>0,\quad x\in\mathbb{R}^n, \tag{1.1}$$

where $a,b\in\mathbb{R}^n$, $c_0\in\mathbb{R}$, and V is either zero or a smooth periodic function on $\mathbb{T}^n$. The term $-a\cdot Du$ is a linear drift in the gradient variable, which becomes the dual variable after the Legendre transform, while $b\cdot x$ is a linear drift in the physical variable. The central point is that these two drifts cannot be treated as independent lower-order perturbations. Under the Legendre transform $y = Du(x)$, $f = u^*$, the constant-medium equation becomes, on the gradient image $Du(\mathbb{R}^n)$,

$$\det D^2 f = \exp\{a\cdot y - b\cdot Df + c_0\}.$$

Thus the two drift directions exchange their roles. In the whole-space problem, the relevant invariant is the scalar coupling $a\cdot b$, together with the distinction between a

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genuine gradient drift $a \neq 0$ and the pure spatial-drift case $a = 0$. In the periodic-corrector problem, the scalar coupling is replaced by the matrix compatibility condition $b = Aa$, where A is the quadratic part of the entire solution.

The starting point is the classical rigidity theory for the real Monge–Ampère equation. The classical Jörgens–Calabi–Pogorelov theorem asserts that every entire smooth strictly convex solution of

$$\det D^2 u = \text{constant} \quad \text{in } \mathbb{R}^n$$

is quadratic; see [Jör54, Cal58, Pog72]. Cheng and Yau [CY86] gave an analytic proof in their work on complete affine hypersurfaces, and Caffarelli and Li [CL03] extended the theory to viscosity solutions and exterior-domain problems. Existence and qualitative questions for entire equations with nonconstant prescribed densities were also studied by Chou and Wang [CW96]. For systematic treatments of the real Monge–Ampère equation and its linearized counterpart, see [Gut16, Fig17, Le24].

A different but closely related rigidity problem arises in affine Kähler–Ricci flat geometry. After a Legendre transform, the potential equation takes the form

$$\det D^2 u = \exp\{-d \cdot Du - d_0\}, \quad x \in \mathbb{R}^n,$$

where $d \in \mathbb{R}^n$ and $d_0 \in \mathbb{R}$ are constants. Li and Xu [LX09] proved that no entire smooth strictly convex solution exists when $d \neq 0$; equivalently, the only entire smooth strictly convex solutions in this class occur in the constant-density case. Xu and Zhu [XZ16] later gave a shorter proof of this result. These results show that a linear gradient drift is not an innocuous perturbation of the constant-density equation. For the broader affine Bernstein problem and related Monge–Ampère equations, see [LJSX10, TW00, TW02, TW05, TW08].

Our first goal is to determine what happens when the gradient drift is coupled to a spatial drift. In the constant-medium case $V \equiv 0$, we study

$$\det D^2 u = \exp\{-a \cdot Du + b \cdot x - c_0\}, \quad D^2 u > 0, \quad x \in \mathbb{R}^n.$$

This equation also appears as the potential equation for entire Lagrangian translating solitons in the pseudo-Euclidean space $\mathbb{R}^{2n}_n$, when written in null coordinates. Translating solitons are a central class of self-similar solutions in Lagrangian mean curvature flow; see Joyce–Lee–Tsui [JLT10] for foundational examples in the Euclidean setting. In the pseudo-Euclidean setting, Xu and Huang [XH13] derived the coupled-drift Monge–Ampère equation for spacelike Lagrangian translating gradient graphs and proved rigidity under a completeness assumption. Huang and Xu [HX15] subsequently obtained further Bernstein-type results for the same equation. Xu and Zhu [XZ15] classified the one-dimensional entire solutions and proved higher-dimensional quadratic rigidity under an asymptotic decay condition on the Hessian. In the broader category of complete spacelike translating submanifolds, Chen and Qiu [CQ16] established nonexistence results by means of a generalized Omori–Yau maximum principle, while Xu and Liu [XL19] classified complete spacelike translators and showed that they are affine planes. These completeness results concern a geometrically stronger class and do not by themselves classify arbitrary entire gradient graphs without a completeness hypothesis.

In the translating-soliton interpretation, (a, b) is the translating vector, and the sign of $a \cdot b$ corresponds to its causal type, up to the normalization convention for the null metric. Wu and Xu [WX21] proved nonexistence in the negative-coupling, or timelike, case $a \cdot b < 0$. The borderline lightlike case $a \cdot b = 0$ is more delicate. In dimension two, Song–Wu–Xu [SWX26] proved nonexistence for lightlike translating vectors satisfying $(a_1, a_2) \neq 0$ and $(b_1, b_2) \neq 0$. The principal new contribution of the present paper is a scalar maximum-principle proof of the null obstruction in every dimension $n \geq 2$, together with a direct ODE argument when $n = 1$. The result applies to arbitrary entire smooth strictly convex gradient graphs, includes the pure gradient-drift case $b = 0$, and requires neither completeness nor Hessian decay.

The resulting whole-space solvability alternatives are sharp.

Theorem 1.1 (Whole-space solvability alternatives)**.** *Let $n \geq 1$, $a, b \in \mathbb{R}^n$, and $c_0 \in \mathbb{R}$. Consider*

$$\det D^2 u = \exp\{-a \cdot Du + b \cdot x - c_0\}, \quad D^2 u > 0, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Then the following statements hold.

(i) *If $a = b = 0$, then every entire smooth strictly convex solution of* (1.2) *is a quadratic polynomial.*

(ii) *If $a \neq 0$ and $a \cdot b \leq 0$, then* (1.2) *admits no entire smooth strictly convex solution.*

(iii) *If either $a = 0$ and $b \neq 0$, or $a \cdot b > 0$, then* (1.2) *admits non-quadratic entire smooth strictly convex solutions.*

The new rigidity regime in Theorem 1.1 is the null-coupling case

$$a \neq 0, \quad a \cdot b = 0.$$

Within the class of arbitrary entire smooth strictly convex solutions, Song–Wu–Xu [SWX26] proved the mixed-null nonexistence result in dimension two. Here we remove the dimension restriction and simultaneously include the pure gradient-drift case $b = 0$.

Our proof exploits a scalar structure specific to the coupled-drift equation. A sublevel-set maximum principle first gives a global oscillation bound for $w = a \cdot Du$. Writing

$$P = a^\top D^2 u \, a,$$

we show that, in the null-coupling case, the full drifted linearized operator makes w harmonic and yields an exact nonnegative square identity for P. A maximum-principle argument applied to a negative power of P, together with a properness penalization, then gives the contradiction. This argument applies in every dimension $n \geq 2$, while the case $n = 1$ follows from an elementary ODE calculation. Neither part requires completeness of the Hessian metric or uniform ellipticity.

This scalar formulation is adapted to the present equation and differs from the affine-geometric approach used by Li–Xu [LX09] for the pure gradient-drift problem.

The relation between the two approaches is discussed in Remark 2.5. The solvable branches in Theorem 1.1 are obtained from explicit one-dimensional and rank-one constructions.

We next turn to periodic media. Periodic Monge–Ampère equations also arise naturally on compact special affine and Hessian manifolds. The foundational solvability theory in this setting goes back to Cheng and Yau [CY82]. Li [Li90] developed existence and uniqueness results for classes of fully nonlinear Monge–Ampère-type equations on compact manifolds, including further results on flat tori, and Caffarelli and Viaclovsky [CV01] established regularity results on Hessian manifolds.

For the classical undrifted equation, periodicity of the Monge–Ampère density leads to a quadratic-periodic decomposition. Caffarelli and Li [CL04] proved that, for a smooth positive periodic density, every entire smooth strictly convex solution has the form

$$u(x) = \frac{1}{2}x^\top Ax + \ell \cdot x + \psi(x),$$

where $A = A^\top > 0$, $\ell \in \mathbb{R}^n$, and ψ is periodic. Li and Lu [LL22] extended this Liouville theorem to periodic densities satisfying $\log f \in L^\infty$. More recently, Jin–Li–Tran–Tu [JLTT25] treated periodic Monge–Ampère measures which are allowed to be degenerate or singular.

The pair consisting of a periodic corrector and an additive constant is also standard in periodic homogenization. For the classical framework, see Bensoussan–Lions–Papanicolaou [BLP78]; for the perturbed-test-function method and periodic homogenization of fully nonlinear equations, see Evans [Eva89, Eva92].

For the coupled-drift equation (1.1), the natural cell problem contains the gradient drift at the periodic scale. Given a prescribed quadratic part $A = A^\top > 0$, one is led to

$$\det(A + D^2\psi) = \exp\{-a \cdot D\psi + V - c\}, \quad A + D^2\psi > 0 \quad \text{on } \mathbb{T}^n.$$

The scalar c is an unknown normalizing constant. To compare this problem with the Monge–Ampère theory on compact Hessian manifolds, set

$$\Phi_A(x) = \frac{1}{2}x^\top Ax + \psi(x), \quad p(x) = D\Phi_A(x) = Ax + D\psi(x).$$

Then the cell equation can be rewritten as the weighted Jacobian equation

$$e^{a\cdot p(x)} \det Dp(x) = e^{a\cdot Ax+V(x)-c}. \tag{1.3}$$

The gradient map is equivariant with respect to lattice translations:

$$p(x + k) = p(x) + Ak, \quad k \in \mathbb{Z}^n.$$

Equation (1.3) resembles the weighted Monge–Ampère equations

$$g(D\Phi) \det D^2\Phi = Cf$$

studied by Hultgren and Önnheim [HÖ19] on compact Hessian manifolds through optimal transport. Their formulation on the universal cover uses locally finite measures invariant under the relevant affine group actions. In the present setting, however, the natural weights

$$f(x) = e^{a\cdot Ax+V(x)}, \quad g(y) = e^{a\cdot y}$$

are generally not invariant. Rather, they transform according to the same character:

$$f(x+k) = e^{a\cdot Ak} f(x), \quad g(y+Ak) = e^{a\cdot Ak} g(y).$$

Thus the drifted cell equation does not fall directly under the invariant-measure formulation of [HÖ19]; it may instead be viewed as a character-twisted weighted Monge–Ampère problem.

Our second main result gives existence and uniqueness for this nonlinear drifted cell problem.

Theorem 1.2 (Periodic cell problem). *For every* $A = A^\top > 0$*,* $a \in \mathbb{R}^n$*, and* $V \in C^\infty(\mathbb{T}^n)$*, there exists a unique pair* (ψ_A, c_A)*, with*

$$\psi_A \in C^\infty(\mathbb{T}^n), \quad \int_{\mathbb{T}^n} \psi_A = 0, \quad c_A \in \mathbb{R},$$

such that

$$\det(A + D^2\psi_A) = \exp\{-a\cdot D\psi_A + V - c_A\}, \quad A + D^2\psi_A > 0 \quad \text{on } \mathbb{T}^n.$$

The normalizing constant also records the dependence of the cell problem on the periodic data. It satisfies

$$\min_{\mathbb{T}^n} V - \log\det A \le c_A \le \max_{\mathbb{T}^n} V - \log\det A.$$

Moreover, for fixed A and a, it is 1-Lipschitz with respect to V in the L^∞ norm, while for fixed a and V, it is monotone decreasing with respect to A in the Loewner order. These properties follow directly from comparison at maximum and minimum points and give a quantitative description of the effective constant.

The proof of Theorem 1.2 uses the continuity method. Periodic semiconvexity supplies uniform C^0 and C^1 bounds, while a maximum-eigenvalue estimate for the full drifted linearized operator controls $A + D^2\psi$. Evans–Krylov and Schauder estimates complete the argument.

The cell problem also identifies the compatibility condition for quadratic-periodic entire solutions. Substituting

$$u(x) = \frac{1}{2}x^\top Ax + \ell\cdot x + \psi(x) + C$$

into (1.1), the nonperiodic part of the exponent is $(b - Aa)\cdot x$. It follows that such a solution exists precisely when

$$b = Aa, \quad c_0 + a\cdot\ell = c_A,$$

in which case ψ agrees with ψ_A up to an additive constant.

For prescribed densities that converge to periodic data at infinity, Teixeira and Zhang [TZ16] proved quadratic-periodic asymptotic structure for global convex solutions in dimensions $n \geq 3$. More recently, Qi and Bao [QB25] studied optimal asymptotic expansions for Hölder perturbations of periodic data. In contrast with these prescribed-density problems, the density in (1.1) depends on the unknown gradient, and this self-consistent dependence produces the compatibility condition $b = Aa$.

Our final result shows that the compatibility condition remains necessary under a subquadratic asymptotic assumption and that a bounded remainder is necessarily the periodic corrector.

Theorem 1.3 (Asymptotically quadratic compatibility and bounded-corrector rigidity)**.** *Let $V \in C^\infty(\mathbb{T}^n)$, and let $u \in C^\infty(\mathbb{R}^n)$ be a strictly convex solution of*

$$\det D^2 u = \exp\{-a \cdot Du + b \cdot x + V(x) - c_0\} \quad \textit{in } \mathbb{R}^n.$$

Suppose that, for some $A = A^\top > 0$ and $\ell \in \mathbb{R}^n$,

$$f(x) = u(x) - \frac{1}{2} x^\top A x - \ell \cdot x$$

satisfies

$$\sup_{B_R} |f| = o(R^2) \quad \textit{as } R \to \infty.$$

Then $b = Aa$.

If, in addition, $f \in L^\infty(\mathbb{R}^n)$, then

$$c_0 + a \cdot \ell = c_A$$

and

$$u(x) = \frac{1}{2} x^\top A x + \ell \cdot x + \psi_A(x) + C$$

for some $C \in \mathbb{R}$.

The compatibility conclusion compares the exponential volume growth caused by $b - Aa \neq 0$ with the polynomial growth of $Du(B_R)$. For bounded remainders, compactness of integer translates and the strong comparison principle force periodicity, after which uniqueness of the cell problem gives the classification.

The paper is organized as follows. Section 2 proves the whole-space alternatives. Section 3 treats the periodic cell problem, and Section 4 proves the compatibility and bounded-corrector rigidity results.

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2. Proof of the whole-space alternatives

We prove Theorem 1.1 for the constant-medium equation

$$\det D^2u = \exp\{-a\cdot Du + b\cdot x - c_0\}, \quad D^2u > 0, \quad x\in\mathbb{R}^n. \tag{2.1}$$

Throughout this section, strict convexity is understood in the classical pointwise sense $D^2u > 0$, as required in (2.1). The proof is organized according to the three regimes identified in the statement of the theorem. We first treat the classical constant-density branch and recall the known obstruction in the negative-coupling case. We then establish the null-coupling obstruction by combining a global oscillation estimate for $a\cdot Du$ with a scalar maximum-principle argument. Finally, we construct explicit non-quadratic entire solutions in the two remaining solvable regimes. This separates the rigidity mechanisms from the constructive part of the classification and makes the sharp transition at $a\cdot b = 0$ transparent.

We begin with the drift-free case. If $a = b = 0$, then (2.1) reduces to the constant-density Monge–Ampère equation

$$\det D^2u = e^{-c_0} \quad \text{in } \mathbb{R}^n.$$

The Jörgens–Calabi–Pogorelov theorem therefore implies that every entire smooth strictly convex solution is a quadratic polynomial; see [Jör54, Cal58, Pog72]. Its constant Hessian matrix necessarily satisfies $\det D^2u = e^{-c_0}$. This gives the rigid branch in Theorem 1.1 and serves as the natural baseline for the analysis of the two drift terms.

Next, we recall the negative-coupling obstruction. Before doing so, we record an elementary consequence of strict convexity that will be used repeatedly.

Lemma 2.1 (Properness after affine normalization)**.** *Let $u\in C^\infty(\mathbb{R}^n)$ be strictly convex and assume that $Du(0) = 0$. Then*

$$u(x)\to+\infty \quad \textit{as } |x|\to\infty.$$

In particular, every sublevel set of u is bounded.

Proof. For $e\in\mathbb{S}^{n-1}$, set

$$g_e(t) = u(te).$$

Then

$$g_e''(t) = e^\top D^2u(te)e > 0, \quad g_e'(0) = 0.$$

Hence $g_e'(1) > 0$ for every $e\in\mathbb{S}^{n-1}$. Since

$$e\longmapsto g_e'(1) = Du(e)\cdot e$$

is continuous on the compact unit sphere,

$$\sigma := \min_{|e|=1} g_e'(1) > 0.$$

For $r\geq 1$, convexity gives

$$u(re) = g_e(r) \geq g_e(1) + (r-1)g_e'(1) \geq \min_{|e|=1} u(e) + \sigma(r-1).$$

The right-hand side tends to $+\infty$ uniformly in e. □

Proposition 2.2 (Negative-coupling obstruction). *Assume that $a \neq 0$ and $a \cdot b < 0$. Then (2.1) admits no entire smooth strictly convex solution.*

Proof. This is the nonexistence theorem of Wu–Xu [WX21, Theorem 1.3]. For completeness, we include an adaptation of their maximum-principle argument, since the same auxiliary function will also be used in the borderline case.

Suppose that a solution exists. After translating the base point, subtracting a supporting affine function, and changing the constant c_0, we may assume

$$u(0) = 3, \quad Du(0) = 0, \quad u > 0.$$

For $C > 3$, set

$$\Omega_C = \{u < C\}, \quad \eta = C - u, \quad \kappa = 3C.$$

By Lemma 2.1, the domain Ω_C is bounded.

Define

$$Q = \exp\left(-\frac{\kappa}{\eta}\right) e^{-b\cdot x} \det D^2 u = \exp\left(-\frac{\kappa}{\eta} - a \cdot Du - c_0\right) \quad \text{in } \Omega_C.$$

The function Q extends continuously by zero to $\partial\Omega_C$. It therefore attains its maximum at an interior point $x_C \in \Omega_C$.

At x_C,

$$0 = (\log Q)_i = -\frac{\kappa u_i}{\eta^2} - a_\ell u_{\ell i}. \tag{2.2}$$

Contracting (2.2) with a_i and with $u^{ij}u_j$, respectively, gives

$$a^\top D^2 u\, a = -\frac{\kappa}{\eta^2} a \cdot Du, \quad -a \cdot Du = \frac{\kappa}{\eta^2} u^{ij} u_i u_j \geq 0. \tag{2.3}$$

At the same point,

$$\begin{aligned} 0 &\geq u^{ij} (\log Q)_{ij} \\ &= a^\top D^2 u\, a - a \cdot b - \frac{\kappa n}{\eta^2} - \frac{2\kappa}{\eta^3} u^{ij} u_i u_j. \end{aligned}$$

Using (2.3), we obtain

$$0 \geq \left(\frac{\kappa}{\eta^2} - \frac{2}{\eta}\right)(-a \cdot Du) - a \cdot b - \frac{\kappa n}{\eta^2}. \tag{2.4}$$

Since $0 < u < C$ and $\kappa = 3C$,

$$\frac{\kappa}{\eta^2} - \frac{2}{\eta} = \frac{\kappa - 2\eta}{\eta^2} = \frac{C + 2u}{(C - u)^2} > 0.$$

The first term in (2.4) is therefore nonnegative. Since $a \cdot b < 0$, it follows that

$$\eta(x_C)^2 \leq \frac{\kappa n}{-a \cdot b} = \frac{3Cn}{-a \cdot b}.$$

Consequently,

$$\frac{\kappa}{\eta(x_C)} \geq c_1\sqrt{C}$$

for some $c_1 > 0$ depending only on n and $-a \cdot b$.

Set

$$X = -a \cdot Du(x_C) \geq 0.$$

Equation (2.4) also gives

$$\left(\frac{\kappa}{\eta(x_C)^2} - \frac{2}{\eta(x_C)}\right) X \leq a \cdot b + \frac{\kappa n}{\eta(x_C)^2} < \frac{\kappa n}{\eta(x_C)^2}.$$

Thus

$$X < \frac{\kappa n}{\kappa - 2\eta(x_C)} < 3n.$$

It follows that

$$\begin{aligned} \log Q(x_C) &= -\frac{\kappa}{\eta(x_C)} - a \cdot Du(x_C) - c_0 \\ &\leq -c_1\sqrt{C} + 3n - c_0, \end{aligned}$$

and therefore

$$Q(x_C) \to 0 \quad \text{as } C \to \infty.$$

On the other hand,

$$Q(0) = \exp\left(-\frac{3C}{C-3} - c_0\right) \to e^{-3-c_0} > 0.$$

For sufficiently large C, this contradicts $Q(0) \leq Q(x_C)$. □

We next turn to the null-coupling case. The proof has two ingredients. The first is a global oscillation estimate for the scalar drift $a \cdot Du$.

Lemma 2.3 (Borderline oscillation)**.** *Let u solve (2.1) and suppose that $a \cdot b = 0$. Then*

$$|a \cdot Du(x) - a \cdot Du(y)| \leq 3n + 3 \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Proof. Fix a base point and perform an affine normalization so that

$$u(0) = 3, \quad Du(0) = 0, \quad u > 0.$$

For $C > 3$, use the same quantities

$$\Omega_C = \{u < C\}, \quad \eta = C - u, \quad \kappa = 3C,$$

and the same auxiliary function Q as in the proof of Proposition 2.2. Let x_C be its interior maximum point.

Equations (2.3) and (2.4) remain valid. Since $a \cdot b = 0$, we obtain

$$0 \geq \left(\frac{\kappa}{\eta^2} - \frac{2}{\eta}\right)(-a \cdot Du) - \frac{\kappa n}{\eta^2}$$

at x_C. Therefore
$$-a \cdot Du(x_C) \le \frac{\kappa n}{\kappa - 2\eta(x_C)} < 3n.$$
For any fixed $x \in \mathbb{R}^n$, once $C > u(x)$, the inequality $Q(x) \le Q(x_C)$ gives
$$-a \cdot Du(x) \le -a \cdot Du(x_C) + \frac{\kappa}{C - u(x)} - \frac{\kappa}{\eta(x_C)}.$$
Discarding the final nonpositive term and letting $C \to \infty$, we find
$$-a \cdot Du(x) \le 3n + 3.$$

To compare two arbitrary points, fix $x_0 \in \mathbb{R}^n$ and consider
$$v_{x_0}(y) = u(x_0 + y) - u(x_0) - Du(x_0) \cdot y + 3.$$
By convexity,
$$v_{x_0}(0) = 3, \quad Dv_{x_0}(0) = 0, \quad v_{x_0} \ge 3.$$
Moreover, v_{x_0} satisfies an equation of the form (2.1) with the same vectors a, b, the same coupling $a \cdot b = 0$, and a possibly different constant. Applying the preceding one-sided estimate to v_{x_0} yields
$$a \cdot Du(x_0 + y) - a \cdot Du(x_0) \ge -(3n + 3).$$
Interchanging the two points gives the reverse inequality. □

The second ingredient converts this oscillation control into rigidity. The quantity $P = a^\top D^2 u\, a$ measures the Hessian curvature in the drift direction. Differentiating the equation with the full drifted linearized operator cancels the third-order drift terms and leaves a nonnegative square, which is the source of coercivity.

Proposition 2.4 (Bounded null-drift rigidity)**.** *Let $n \ge 2$, $a \ne 0$, and $a \cdot b = 0$. Suppose that $u \in C^4(\mathbb{R}^n)$ is strictly convex, solves* (2.1)*, and satisfies*
$$\operatorname{osc}_{\mathbb{R}^n}(a \cdot Du) < \infty.$$
Then no such solution exists.

Proof. Suppose, to the contrary, that such a solution exists. Fix a point $x_0 \in \mathbb{R}^n$ and define
$$\widetilde{u}(x) = u(x + x_0) - u(x_0) - Du(x_0) \cdot x.$$
Then
$$\widetilde{u}(0) = 0, \quad D\widetilde{u}(0) = 0, \quad \widetilde{u} \ge 0$$
by convexity. Moreover, $\widetilde{u}$ satisfies an equation of the same form as (2.1), with the same vectors a and b, and with c_0 replaced by
$$\widetilde{c}_0 = c_0 + a \cdot Du(x_0) - b \cdot x_0.$$
Thus the condition $a \cdot b = 0$ is unchanged. Since
$$a \cdot D\widetilde{u}(x) = a \cdot Du(x + x_0) - a \cdot Du(x_0),$$

the oscillation assumption is also unchanged. Renaming $\widetilde{u}$ as u, we may assume

$$u(0)=0, \quad Du(0)=0, \quad u\geq 0.$$

By Lemma 2.1,

$$u(x)\to+\infty \quad \text{as } |x|\to\infty.$$

Set

$$w=a\cdot Du.$$

Since $w(0)=0$, the oscillation assumption implies

$$|w|\leq W \quad \text{in } \mathbb{R}^n$$

for some finite constant W.

Write

$$g_{ij}=u_{ij}, \quad (g^{ij})=(g_{ij})^{-1},$$

and define the drifted linearized operator

$$\mathcal{L}f=g^{ij}f_{ij}+a_kf_k.$$

We use the summation convention and set

$$P=a_ra_su_{rs}=a^\top D^2u\,a>0, \quad T_{ij}=a_ku_{ijk}.$$

Here and below,

$$|T|_g^2=g^{ip}g^{jq}T_{ij}T_{pq}, \quad |\nabla P|_g^2=g^{ij}P_iP_j.$$

It is useful first to record the relation between w and P. Since

$$w_i=a_ku_{ki}=g_{ik}a_k,$$

we have

$$P=g^{ij}w_iw_j=|\nabla w|_g^2.$$

Moreover,

$$w_{ij}=a_ku_{kij}=T_{ij}.$$

Taking the logarithm of (2.1) gives

$$\log\det(g_{ij})=-a\cdot Du+b\cdot x-c_0.$$

Differentiating in the x_k-direction yields

$$g^{ij}u_{ijk}=b_k-a_\ell u_{\ell k}.$$

Contracting with a_k and using $a\cdot b=0$, we obtain

$$g^{ij}T_{ij}=-P. \tag{2.5}$$

Since

$$a\cdot Dw=P,$$

identity (2.5) also gives

$$\mathcal{L}w=g^{ij}w_{ij}+a\cdot Dw=-P+P=0.$$

Differentiating the logarithmic equation once more gives

$$g^{ij}u_{ijrs}-g^{ip}g^{jq}u_{ijr}u_{pqs}=-a_ku_{krs}.$$

Multiplying by $a_r a_s$, we find

$$g^{ij}P_{ij} - g^{ip}g^{jq}T_{ij}T_{pq} = -a_k P_k.$$

Therefore the first-order term in $\mathcal{L}$ cancels the right-hand side, and

$$\mathcal{L}P = g^{ip}g^{jq}T_{ij}T_{pq} = |T|_g^2. \tag{2.6}$$

Differentiating the definition of P also gives

$$P_i = a_r a_s u_{rsi} = T_{ij}a_j. \tag{2.7}$$

We now derive a pointwise coercive inequality. Fix a point. With respect to the inner product g at that point, choose a g-orthonormal basis whose first vector is parallel to a. In this basis,

$$g_{ij} = \delta_{ij}, \quad a = \sqrt{P}\, e_1.$$

This is only a pointwise algebraic choice; no derivatives of the frame are taken. Equations (2.6), (2.5), and (2.7) become

$$\mathcal{L}P = |T|^2, \quad \operatorname{tr} T = -P,$$

and

$$|\nabla P|_g^2 = P\sum_i T_{1i}^2, \quad a \cdot DP = PT_{11}. \tag{2.8}$$

Choose

$$\theta = \frac{1}{4(n-1)}.$$

In particular,

$$0 < \theta < 1.$$

The chain rule, (2.6), and (2.8) give

$$\begin{aligned}\mathcal{L}(-P^{-\theta}) &= \theta P^{-\theta-1}\mathcal{L}P - \theta(\theta+1)P^{-\theta-2}|\nabla P|_g^2 \\ &= \theta P^{-\theta-1}\left(|T|^2 - (\theta+1)\sum_i T_{1i}^2\right).\end{aligned}$$

Write

$$T = \begin{pmatrix} \tau & v^\top \\ v & B \end{pmatrix}, \quad \tau = T_{11},$$

where

$$v = (T_{12}, \dots, T_{1n})$$

and

$$B = (T_{\alpha\beta})_{2\le\alpha,\beta\le n}.$$

Then

$$|T|^2 = \tau^2 + 2|v|^2 + |B|^2$$

and

$$\sum_i T_{1i}^2 = \tau^2 + |v|^2.$$

Consequently,

$$|T|^2 - (\theta+1)\sum_i T_{1i}^2 = -\theta\tau^2 + (1-\theta)|v|^2 + |B|^2$$
$$\geq -\theta\tau^2 + |B|^2.$$

By (2.5),

$$\operatorname{tr} B = -P - \tau.$$

Since B is an $(n-1)\times(n-1)$ symmetric matrix, the Cauchy–Schwarz inequality gives

$$|B|^2 \geq \frac{(\operatorname{tr} B)^2}{n-1} = \frac{(P+\tau)^2}{n-1}.$$

We have therefore proved the pointwise estimate

$$\mathcal{L}(-P^{-\theta}) \geq \theta P^{-\theta-1}\left(\frac{(P+\tau)^2}{n-1} - \theta\tau^2\right). \tag{2.9}$$

For $\varepsilon > 0$, define

$$F_\varepsilon = -P^{-\theta} - \varepsilon u.$$

Since $P > 0$, one has $-P^{-\theta} \leq 0$. Together with $u(x) \to +\infty$, this gives

$$F_\varepsilon(x) \to -\infty \quad \text{as } |x| \to \infty.$$

Thus F_ε attains a global maximum at some point x_ε.

At x_ε,

$$DF_\varepsilon = 0,$$

and hence

$$\theta P^{-\theta-1} DP = \varepsilon Du.$$

Choose at x_ε the adapted g-orthonormal basis used above. Contracting the preceding identity with a and using (2.8), we obtain

$$\theta P^{-\theta-1}(P\tau) = \varepsilon w.$$

Thus

$$\tau = \frac{\varepsilon}{\theta} w P^\theta, \quad |\tau| \leq \frac{\varepsilon W}{\theta} P^\theta. \tag{2.10}$$

All quantities in the rest of the argument are evaluated at x_ε.

Suppose first that

$$|\tau| \geq \frac{P}{2}.$$

Then (2.10) gives

$$\frac{P}{2} \leq \frac{\varepsilon W}{\theta} P^\theta,$$

and hence

$$P^{1-\theta} \leq \frac{2W}{\theta}\varepsilon. \tag{2.11}$$

Suppose instead that

$$|\tau| < \frac{P}{2}.$$

Then

$$P + \tau \geq \frac{P}{2}$$

and

$$\tau^2 < \frac{P^2}{4}.$$

Therefore

$$\begin{aligned} \frac{(P+\tau)^2}{n-1} - \theta\tau^2 &\geq \frac{P^2}{4(n-1)} - \frac{\theta P^2}{4} \\ &= \frac{P^2}{4}\left(\frac{1}{n-1} - \theta\right). \end{aligned}$$

Since

$$\theta = \frac{1}{4(n-1)},$$

the right-hand side equals

$$\frac{3P^2}{16(n-1)} \geq \frac{P^2}{8(n-1)}.$$

Estimate (2.9) consequently gives

$$\mathcal{L}(-P^{-\theta}) \geq \frac{\theta}{8(n-1)} P^{1-\theta}.$$

At the maximum point,

$$DF_\varepsilon(x_\varepsilon) = 0$$

and

$$D^2 F_\varepsilon(x_\varepsilon) \leq 0.$$

The first-order part of $\mathcal{L}$ therefore vanishes there, and

$$\mathcal{L}F_\varepsilon(x_\varepsilon) = g^{ij}(F_\varepsilon)_{ij}(x_\varepsilon) \leq 0.$$

Furthermore,

$$\mathcal{L}u = g^{ij}u_{ij} + a \cdot Du = n + w \leq n + W.$$

It follows that

$$\begin{aligned} 0 \geq \mathcal{L}F_\varepsilon &= \mathcal{L}(-P^{-\theta}) - \varepsilon\mathcal{L}u \\ &\geq \frac{\theta}{8(n-1)} P^{1-\theta} - \varepsilon(n+W). \end{aligned}$$

Hence

$$P^{1-\theta} \leq \frac{8(n-1)(n+W)}{\theta}\varepsilon. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$P(x_\varepsilon)^{1-\theta} \leq C\varepsilon,$$

where C is independent of ε. Since $1-\theta>0$, this implies

$$P(x_\varepsilon)\to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{2.13}$$

On the other hand, maximality of F_ε, together with $u(0)=0$ and $u\geq 0$, gives

$$-P(x_\varepsilon)^{-\theta}\geq -P(x_\varepsilon)^{-\theta}-\varepsilon u(x_\varepsilon)=F_\varepsilon(x_\varepsilon)\geq F_\varepsilon(0)=-P(0)^{-\theta}.$$

Thus

$$P(x_\varepsilon)\geq P(0)>0.$$

This contradicts (2.13). □

Remark 2.5 (Relation with the Li–Xu [LX09] argument). The proof of Proposition 2.4 uses only the bounded oscillation of $a\cdot Du$. In particular, when $b=0$, it gives an independent proof, in the smooth strictly convex setting, of the affine Kähler–Ricci flat rigidity theorem of Li–Xu [LX09, Main Theorem].

The scalar quantity used here is closely related to the affine invariant in the Li–Xu argument. To see this, consider the pure gradient-drift equation and let $f=u^*$ be the Legendre transform of u, defined on the gradient image $Du(\mathbb{R}^n)$. Then

$$\det D^2 f=\exp\{a\cdot y+c_0\}.$$

If

$$\rho=(\det D^2 f)^{-1/(n+2)}$$

and

$$\Phi=\frac{f^{ij}\rho_i\rho_j}{\rho^2},$$

as in [LX09], then

$$\frac{\rho_i}{\rho}=-\frac{a_i}{n+2}.$$

At corresponding dual points $y=Du(x)$, one has

$$D^2 f(y)=(D^2u(x))^{-1},$$

and hence

$$\Phi(y)=\frac{1}{(n+2)^2}a^\top D^2u(x)\,a=\frac{P(x)}{(n+2)^2}.$$

Thus the simplification in the present proof does not come from introducing an unrelated scalar invariant. Rather, it comes from working in the original variables with the full linearized operator

$$\mathcal{L}=u^{ij}\partial_{ij}+a\cdot D$$

and retaining the distinguished direction selected by a. In the null-coupling case, the quantities

$$w=a\cdot Du,\quad P=a^\top D^2u\,a,\quad T_{ij}=a_k u_{ijk}$$

satisfy the closed identities

$$\mathcal{L}w=0,\quad P=|\nabla w|^2_{D^2u},\quad \mathcal{L}P=|T|^2_{D^2u},\quad \operatorname{tr}_{D^2u}T=-P.$$

Together with the global oscillation bound for w, this structure leads to the direct maximum-principle argument above.

Remark 2.6 (Graph-preserving changes of variables). The mixed null-coupling case with $a \neq 0$ and $b \neq 0$ is not reduced to the pure gradient-drift case by the natural changes of variables that preserve the class of entire convex gradient graphs. Indeed, let $S \in GL(n, \mathbb{R})$ and set

$$v(z) = u(Sz) + q \cdot z + d.$$

Then v satisfies an equation of the same form, with transformed drift vectors

$$a' = S^{-1}a, \qquad b' = S^{\top}b, \qquad a' \cdot b' = a \cdot b,$$

and with a modified additive constant. Thus an invertible linear change of the base variables cannot turn a nonzero spatial drift into zero. The Legendre transform exchanges the two nonzero drift vectors and therefore does not remove either one. Moreover, it is defined on the gradient image $Du(\mathbb{R}^n)$, which need not be all of $\mathbb{R}^n$; hence it does not, in general, preserve the class of entire solutions on a fixed Euclidean base.

Ambient pseudo-orthogonal transformations may identify nonzero lightlike translation vectors, but they need not preserve the chosen graph polarization. In particular, the transformed Lagrangian submanifold need not remain an entire strictly convex gradient graph over the same $\mathbb{R}^n$. The mixed null-coupling obstruction is therefore not an immediate consequence of the pure gradient-drift result through transformations that preserve the global graph class considered here.

We now obtain the full null-coupling obstruction.

Proposition 2.7 (Null-coupling obstruction). *Assume that*

$$a \neq 0, \quad a \cdot b = 0.$$

Then (2.1) *admits no entire smooth strictly convex solution.*

Proof. Assume first that $n \geq 2$. Lemma 2.3 gives

$$\operatorname{osc}_{\mathbb{R}^n}(a \cdot Du) < \infty.$$

The conclusion therefore follows from Proposition 2.4.

If $n = 1$, then $a \neq 0$ and $ab = 0$ imply $b = 0$. The equation is

$$u'' = e^{-au' - c_0}.$$

Consequently,

$$\left(e^{au'}\right)' = ae^{-c_0} \neq 0.$$

Thus $e^{au'}$ is a nonconstant affine function of x, which cannot remain positive on all of $\mathbb{R}$. This is a contradiction. ∎

We now construct non-quadratic entire solutions in the remaining two regimes.

Suppose first that $a = 0$ and $b \neq 0$. After an orthogonal change of coordinates, write

$$b = \beta e_n, \quad \beta = |b| > 0.$$

Then

$$u(x', x_n) = \frac{1}{2}|x'|^2 + \frac{e^{-c_0}}{\beta^2} e^{\beta x_n}$$

is smooth and strictly convex, and

$$\det D^2 u = e^{\beta x_n - c_0} = e^{b \cdot x - c_0}.$$

It is plainly non-quadratic.

Suppose next that

$$a \cdot b > 0.$$

We first choose a symmetric positive definite matrix A satisfying

$$Aa = b.$$

For $n = 1$, simply take $A = b/a > 0$. For $n \geq 2$, use an orthonormal basis whose first vector is $e_1 = a/|a|$, and write

$$\frac{b}{|a|} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha = \frac{a \cdot b}{|a|^2} > 0, \quad \beta \in \mathbb{R}^{n-1}.$$

For any

$$\lambda > \frac{|\beta|^2}{\alpha},$$

the matrix

$$A = \begin{pmatrix} \alpha & \beta^\top \\ \beta & \lambda I_{n-1} \end{pmatrix}$$

is positive definite by the Schur complement criterion and satisfies $Aa = b$.

The same compatibility already produces quadratic solutions. Indeed, since $a \neq 0$, one may choose $\ell_0 \in \mathbb{R}^n$ so that

$$a \cdot \ell_0 = -c_0 - \log \det A.$$

Then

$$u_0(x) = \frac{1}{2} x^\top A x + \ell_0 \cdot x$$

solves (2.1). To show that the positive-coupling regime also contains non-quadratic solutions, choose $\ell \in \mathbb{R}^n$ such that

$$\gamma = a \cdot \ell \neq 0,$$

and set

$$\mu = \ell^\top A^{-1} \ell > 0, \quad K_0 = \frac{e^{-c_0}}{\det A}.$$

Fix $C_0 > 0$, and define a smooth function H on $\mathbb{R}$, up to an additive constant, by

$$e^{\gamma H'(t)} = K_0 + C_0 e^{-\gamma t/\mu}.$$

Differentiating gives

$$H''(t) = -\frac{1}{\mu} \frac{C_0 e^{-\gamma t/\mu}}{K_0 + C_0 e^{-\gamma t/\mu}},$$

and hence

$$1 + \mu H''(t) = \frac{K_0}{K_0 + C_0 e^{-\gamma t/\mu}} = K_0 e^{-\gamma H'(t)} > 0. \tag{2.14}$$

Define

$$u(x) = \frac{1}{2} x^\top A x + H(\ell \cdot x).$$

Then

$$D^2 u = A + H''(\ell \cdot x)\, \ell \otimes \ell.$$

Since $A > 0$, the rank-one update criterion and (2.14) imply that $D^2 u > 0$. The matrix determinant lemma gives

$$\det D^2 u = \det A \left(1 + \mu H''(\ell \cdot x)\right).$$

Moreover, since A is symmetric and $Aa = b$,

$$\begin{aligned} -a \cdot Du + b \cdot x &= -a \cdot \left(Ax + H'(\ell \cdot x)\ell\right) + b \cdot x \\ &= -\gamma H'(\ell \cdot x). \end{aligned}$$

Using (2.14) and the definition of K_0, we conclude that

$$\begin{aligned} \det D^2 u &= \det A\, K_0 e^{-\gamma H'(\ell \cdot x)} \\ &= \exp\{-a \cdot Du + b \cdot x - c_0\}. \end{aligned}$$

Because $C_0 > 0$, the function H is not quadratic, and neither is u.

Remark 2.8 (Degeneration at the null threshold). The positive-coupling construction cannot remain uniformly convex as $a \cdot b \downarrow 0$ with $a \neq 0$. Indeed, every positive definite matrix A satisfying $Aa = b$ obeys

$$a^\top A a = a \cdot b, \quad \lambda_{\min}(A) \leq \frac{a \cdot b}{|a|^2}.$$

Thus the quadratic background necessarily degenerates at the null threshold, consistent with the nonexistence result there. In particular, the positive and null branches are not connected through a uniformly convex family of these explicit solutions.

Proof of Theorem 1.1. If $a = b = 0$, the classical Jörgens–Calabi–Pogorelov theorem gives part (i).

Suppose that $a \neq 0$ and $a \cdot b \leq 0$. If $a \cdot b < 0$, Proposition 2.2 gives nonexistence. If $a \cdot b = 0$, Proposition 2.7 gives nonexistence. This proves part (ii).

The two explicit constructions above prove part (iii) and complete the proof. □

3. The periodic cell problem

Throughout this section, $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is normalized to have volume one. Let

$$A = A^\top > 0, \quad a \in \mathbb{R}^n, \quad V \in C^\infty(\mathbb{T}^n).$$

We prove existence and uniqueness of a constant $c_A \in \mathbb{R}$ and a zero-average function $\psi_A \in C^\infty(\mathbb{T}^n)$ satisfying

$$\det(A + D^2\psi_A) = \exp\{-a \cdot D\psi_A + V - c_A\}, \quad A + D^2\psi_A > 0 \quad \text{on } \mathbb{T}^n. \tag{3.1}$$

We also record comparison and stability properties of the normalizing constant. The notation suppresses the remaining data: more explicitly, $c_A = c(A; a, V)$.

We use the continuity path

$$\log\det(A + D^2\psi) + t\, a \cdot D\psi - tV + c = 0, \quad t \in [0, 1]. \tag{3.2}$$

The main point is a uniform C^2 bound: in the full linearized operator, the drift cancels the third-order drift contribution in the maximum-eigenvalue estimate.

Remark 3.1 (Relation with compact Hessian-manifold formulations). The weighted Jacobian form in (1.3) has a natural equivariant interpretation. With

$$f(x) = e^{a\cdot Ax + V(x)}, \qquad g(y) = e^{a\cdot y}, \qquad \chi(k) = e^{a\cdot Ak},$$

the source and target weights satisfy, for every $k \in \mathbb{Z}^n$,

$$f(x + k) = \chi(k)f(x), \qquad g(y + Ak) = \chi(k)g(y).$$

Equivalently, they can be viewed as equivariant densities, or as sections of flat real line bundles determined by the same character. The optimal-transport theory of Hultgren–Önnheim [HÖ19], however, is formulated for ordinary invariant measures on the relevant affine quotients, so it does not directly provide the present cell solution.

Guedj–Tô [GT24] developed comparison and existence theory for twisted Monge–Ampère operators on compact Hessian manifolds of the form

$$M_{\rho,g}[\varphi] = \rho \det(g + D^2\varphi)\, dx,$$

where ρ is a fixed density and the right-hand side may depend on x and φ. In contrast, the weight in (3.1) depends on the unknown gradient through $e^{-a\cdot D\psi}$. Thus the existing invariant-measure and fixed-density twisted theories are closely related to, but do not directly subsume, the drifted cell problem considered here.

Comparison, uniqueness, and the normalizing constant. We begin with the comparison principle used both here and in the bounded-corrector classification.

Lemma 3.2 (Strong comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be connected, let $N = N^\top$ be a smooth matrix field on Ω, and let $u_1, u_2 \in C^2(\Omega)$ satisfy*

$$N + D^2u_i > 0, \quad i = 1, 2.$$

Assume that

$$\log\det(N + D^2u_1) + a \cdot Du_1 = \log\det(N + D^2u_2) + a \cdot Du_2 \quad \text{in } \Omega.$$

If

$$u_1 \le u_2 \quad \text{in } \Omega$$

and equality holds at an interior point, then

$$u_1 \equiv u_2 \quad \text{in } \Omega.$$

Proof. Set

$$w = u_1 - u_2.$$

Subtracting the two equations and integrating the logarithmic determinant along the segment $u_2 + sw$, $s \in [0,1]$, gives

$$B^{ij} w_{ij} + a \cdot Dw = 0,$$

where

$$B^{ij}(x) = \int_0^1 \left(N + D^2 u_2 + sD^2 w\right)^{-1}_{ij} \, \mathrm{d}s.$$

The positive definite cone is convex, so every matrix inside the integral is positive definite. Thus the resulting linear operator is uniformly elliptic on compact subsets of Ω. Since $w \le 0$ and w attains an interior maximum, the strong maximum principle gives $w \equiv 0$. ☐

Lemma 3.3 (Uniqueness for the cell problem)**.** *Suppose that (ψ_1, c_1) and (ψ_2, c_2) are two smooth solutions of (3.1). Then $c_1 = c_2$, and $\psi_1 - \psi_2$ is constant. In particular, if both functions have zero average, then $\psi_1 = \psi_2$.*

Proof. Let

$$w = \psi_1 - \psi_2.$$

At a maximum point x_+ of w,

$$Dw(x_+) = 0, \quad D^2 w(x_+) \le 0.$$

Consequently,

$$A + D^2\psi_1(x_+) \le A + D^2\psi_2(x_+).$$

The determinant is monotone on the positive definite cone, and $D\psi_1(x_+) = D\psi_2(x_+)$. Evaluating the equations at x_+ therefore gives $c_1 \ge c_2$. Applying the same argument at a minimum point of w gives $c_1 \le c_2$. Hence $c_1 = c_2$.

Let

$$m = \max_{\mathbb{T}^n} (\psi_1 - \psi_2).$$

The equation is invariant under adding constants to ψ, so $\psi_2 + m$ satisfies the same equation as ψ_2. Moreover,

$$\psi_1 \le \psi_2 + m$$

and equality holds at an interior point of the periodic lift to $\mathbb{R}^n$. Lemma 3.2 gives

$$\psi_1 \equiv \psi_2 + m.$$

If both functions have zero average, then $m = 0$. ☐

The normalizing constant satisfies bounds and comparison properties that are independent of the existence argument.

Lemma 3.4 (Bounds and comparison for the normalizing constant)**.** *The following statements hold.*

(i) *Suppose that* (ψ, c) *solves*

$$\det(A + D^2\psi) = e^{-ta\cdot D\psi + tV - c}, \quad A + D^2\psi > 0,$$

for some $t \in [0,1]$. *Then*

$$t\min_{\mathbb{T}^n} V - \log\det A \le c \le t\max_{\mathbb{T}^n} V - \log\det A. \tag{3.3}$$

(ii) *For fixed* A *and* a, *suppose that* (ψ_i, c_i) *solves the cell problem with potential* V_i, $i = 1, 2$. *Then*

$$\min_{\mathbb{T}^n}(V_1 - V_2) \le c_1 - c_2 \le \max_{\mathbb{T}^n}(V_1 - V_2). \tag{3.4}$$

In particular,

$$|c_1 - c_2| \le \|V_1 - V_2\|_{L^\infty(\mathbb{T}^n)}. \tag{3.5}$$

(iii) *Fix* a *and* V. *Suppose that* $A_1, A_2 > 0$ *are symmetric and*

$$A_1 \le A_2$$

in the sense of quadratic forms. If c_{A_i} *denotes the corresponding normalizing constant, then*

$$c_{A_1} \ge c_{A_2}.$$

If $A_2 - A_1 > 0$, *then the inequality is strict.*

Proof. Let x_+ be a maximum point of ψ. Then

$$D\psi(x_+) = 0, \quad D^2\psi(x_+) \le 0.$$

Thus

$$A + D^2\psi(x_+) \le A,$$

and hence

$$e^{tV(x_+)-c} = \det(A + D^2\psi(x_+)) \le \det A.$$

Therefore

$$c \ge tV(x_+) - \log\det A \ge t\min_{\mathbb{T}^n} V - \log\det A.$$

The upper bound follows similarly by evaluating at a minimum point of ψ, where $D^2\psi \ge 0$. This proves (3.3).

For the second statement, let x_+ be a maximum point of $\psi_1 - \psi_2$. At x_+,

$$D\psi_1 = D\psi_2, \quad D^2\psi_1 \le D^2\psi_2.$$

It follows that

$$V_1(x_+) - c_1 \le V_2(x_+) - c_2,$$

and hence

$$c_1 - c_2 \ge V_1(x_+) - V_2(x_+) \ge \min_{\mathbb{T}^n}(V_1 - V_2).$$

Applying the same argument at a minimum point of $\psi_1 - \psi_2$ gives the opposite bound in (3.4). Estimate (3.5) follows immediately.

Finally, let ψ_i be the solution corresponding to A_i, and let x_+ be a maximum point of $\psi_1 - \psi_2$. Then

$$D^2\psi_1(x_+) \leq D^2\psi_2(x_+),$$

so

$$A_1 + D^2\psi_1(x_+) \leq A_2 + D^2\psi_2(x_+).$$

Comparing the two equations at x_+ gives $c_{A_1} \geq c_{A_2}$. If $A_2 - A_1 > 0$, the matrix inequality at x_+ is strict, and the strict monotonicity of the determinant on the positive definite cone gives $c_{A_1} > c_{A_2}$. □

For fixed A and a, we write $c_A[V]$ for the normalizing constant associated with the potential V. As immediate consequences of Lemma 3.4, one has

$$V_1 \leq V_2 \quad \Longrightarrow \quad c_A[V_1] \leq c_A[V_2],$$

and

$$c_A[V + \lambda] = c_A[V] + \lambda \quad \text{for every constant } \lambda \in \mathbb{R}.$$

***A priori* estimates.** For $t \in [0, 1]$, suppose that (ψ, c) is a smooth solution of (3.2), normalized by

$$\int_{\mathbb{T}^n} \psi = 0.$$

Set

$$M = A + D^2\psi.$$

All estimates below are independent of t. We begin with the elementary periodic semiconvexity bounds.

Lemma 3.5 (Periodic semiconvexity estimate)**.** *There exists a constant $C = C(A, n)$ such that every solution along the continuity path satisfies*

$$\|D\psi\|_{L^\infty(\mathbb{T}^n)} \leq C \tag{3.6}$$

and

$$\|\psi\|_{L^\infty(\mathbb{T}^n)} \leq C. \tag{3.7}$$

Proof. Since

$$A + D^2\psi > 0,$$

we have

$$D^2\psi \geq -A$$

as quadratic forms. In particular,

$$\psi_{ii} \geq -A_{ii}.$$

Fix all variables except x_i, and let

$$f(s) = \psi(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n).$$

Then f is 1-periodic and satisfies

$$f'' \geq -A_{ii}.$$

We claim that

$$\|f'\|_{L^\infty([0,1])} \le A_{ii}.$$

Indeed, for $0 \le s < t \le 1$,

$$f'(t) - f'(s) \ge -A_{ii}(t-s).$$

Applying the same inequality to the periodically extended function on the interval $[t, s+1]$ gives

$$f'(s) - f'(t) \ge -A_{ii}(s+1-t).$$

Hence

$$\operatorname{osc}_{[0,1]} f' \le A_{ii}.$$

Since

$$\int_0^1 f'(s)\,\mathrm{d}s = 0,$$

the asserted bound follows.

Applying this argument in each coordinate direction gives the explicit bound

$$\|D\psi\|_{L^\infty(\mathbb{T}^n)} \le \left(\sum_{i=1}^n A_{ii}^2\right)^{1/2} \le \sqrt{n}\,\lambda_{\max}(A),$$

and hence (3.6). Since ψ has zero average, its minimum is nonpositive and its maximum is nonnegative. The gradient bound therefore implies

$$\operatorname{osc}_{\mathbb{T}^n} \psi \le \operatorname{diam}(\mathbb{T}^n)\|D\psi\|_{L^\infty(\mathbb{T}^n)},$$

which gives (3.7). □

We next record the determinant identity used to control the normalizing constant.

Lemma 3.6 (Periodic determinant identity)**.** *For every smooth periodic ψ,*

$$\int_{\mathbb{T}^n} \det(A + D^2\psi)\,\mathrm{d}x = \det A.$$

Proof. For $s \in [0,1]$, define

$$I(s) = \int_{\mathbb{T}^n} \det(A + sD^2\psi)\,\mathrm{d}x.$$

Since

$$A + sD^2\psi = D^2\left(\frac{1}{2}x^\top Ax + s\psi\right),$$

its cofactor matrix is divergence-free:

$$\partial_i \operatorname{cof}(A + sD^2\psi)^{ij} = 0.$$

Therefore

$$\begin{aligned} I'(s) &= \int_{\mathbb{T}^n} \operatorname{cof}(A + sD^2\psi)^{ij}\psi_{ij}\,\mathrm{d}x \\ &= \int_{\mathbb{T}^n} \partial_i \left(\operatorname{cof}(A + sD^2\psi)^{ij}\psi_j\right)\,\mathrm{d}x \\ &= 0. \end{aligned}$$

Thus $I(1) = I(0) = \det A$. □

Combining (3.2) with Lemma 3.6, we obtain the exact identity

$$c = \log\left(\frac{\displaystyle\int_{\mathbb{T}^n} e^{-ta\cdot D\psi + tV}\,\mathrm{d}x}{\det A}\right).$$

For the *a priori* estimate, however, the sharper pointwise bounds (3.3) are more useful. Together with Lemma 3.5, they imply

$$|c| \le C(A, V).$$

The equation

$$\det M = e^{-ta\cdot D\psi + tV - c}$$

then yields

$$0 < \lambda \le \det M \le \Lambda < \infty, \tag{3.8}$$

where λ, Λ depend only on A, a, V, and n.

We now establish the principal second-derivative estimate.

Proposition 3.7 (Uniform second-derivative estimate)**.** *There exists a constant* $C = C(A, a, n, \|V\|_{C^2(\mathbb{T}^n)})$ *such that every solution of* (3.2) *satisfies*

$$C^{-1}I \le A + D^2\psi \le CI \quad \text{on } \mathbb{T}^n. \tag{3.9}$$

Proof. Define the full linearized operator

$$\mathcal{L}_t = M^{ij}\partial_{ij} + t\,a\cdot D.$$

Fix a constant $B > 0$, to be chosen below, and consider

$$G(x,\xi) = \log M_{\xi\xi}(x) - B\psi(x), \quad (x,\xi) \in \mathbb{T}^n \times \mathbb{S}^{n-1}.$$

Let (x_0, ξ_0) be a maximum point of G. After a constant orthogonal change of coordinates, we may assume that at x_0,

$$\xi_0 = e_1$$

and

$$M(x_0) = \operatorname{diag}(M_1, \dots, M_n), \quad M_1 = M_{11} = \lambda_{\max}(M(x_0)).$$

All computations below are performed at x_0.

Differentiating (3.2) twice in the x_1-direction gives

$$M^{ij}M_{ij,11} - M^{ip}M^{jq}M_{ij,1}M_{pq,1} + ta_kM_{11,k} = tV_{11}.$$

Since

$$M_{ij,11} = M_{11,ij},$$

we obtain

$$M^{ij}(\log M_{11})_{ij} = \frac{M^{ip}M^{jq}M_{ij,1}M_{pq,1}}{M_{11}} - \frac{M^{ij}M_{11,i}M_{11,j}}{M_{11}^2} - \frac{ta_k M_{11,k}}{M_{11}} + \frac{tV_{11}}{M_{11}}.$$

At the diagonal point, the difference of the two third-order quadratic expressions is nonnegative. The full symmetry of the third derivatives gives

$$M_{11,i} = M_{1i,1}.$$

Consequently,

$$\frac{M^{ip}M^{jq}M_{ij,1}M_{pq,1}}{M_{11}} - \frac{M^{ij}M_{11,i}M_{11,j}}{M_{11}^2} = \sum_{i>1} \frac{M_{1i,1}^2}{M_1^2 M_i} + \sum_{i,j>1} \frac{M_{ij,1}^2}{M_i M_j M_1} \geq 0.$$

Consequently,

$$M^{ij}(\log M_{11})_{ij} \geq -\frac{ta_k M_{11,k}}{M_{11}} + \frac{tV_{11}}{M_{11}}.$$

The drift part of $\mathcal{L}_t$ cancels the third-order drift term:

$$t\, a \cdot D \log M_{11} = \frac{ta_k M_{11,k}}{M_{11}}.$$

Thus

$$\mathcal{L}_t \log M_{11} \geq \frac{tV_{11}}{M_{11}}. \tag{3.10}$$

On the other hand,

$$\mathcal{L}_t(-B\psi) = -BM^{ij}\psi_{ij} - Bt\, a \cdot D\psi = -Bn + B\operatorname{tr}(M^{-1}A) - Bt\, a \cdot D\psi,$$

because

$$M^{ij}\psi_{ij} = M^{ij}(M_{ij} - A_{ij}) = n - \operatorname{tr}(M^{-1}A).$$

Since $G(\,\cdot\,, e_1)$ has a maximum at x_0,

$$\mathcal{L}_t G(x_0, e_1) \leq 0.$$

Combining this with (3.10), we obtain

$$0 \geq B\operatorname{tr}(M^{-1}A) - Bn - Bt\, a \cdot D\psi + \frac{tV_{11}}{M_{11}}. \tag{3.11}$$

Let

$$\alpha = \lambda_{\min}(A) > 0, \qquad K_V = \sup_{x\in\mathbb{T}^n} \|D^2V(x)\|_{\mathrm{op}}.$$

Then

$$\operatorname{tr}(M^{-1}A) \geq \alpha \operatorname{tr}(M^{-1}) \geq \frac{\alpha}{M_{11}}.$$

It follows that

$$\frac{tV_{11}}{M_{11}} \geq -\frac{K_V}{\alpha} \operatorname{tr}(M^{-1}A).$$

Using the C^1 bound from Lemma 3.5, inequality (3.11) gives

$$0 \geq \left(B - \frac{K_V}{\alpha}\right) \operatorname{tr}(M^{-1}A) - B\left(n + |a| \|D\psi\|_{L^\infty}\right).$$

Choose

$$B = 1 + \frac{K_V}{\alpha}.$$

Then

$$\operatorname{tr}(M(x_0)^{-1}A) \leq C.$$

Equivalently,

$$A^{1/2}M(x_0)^{-1}A^{1/2} \leq CI,$$

and hence

$$M(x_0) \geq C^{-1}A \geq C^{-1}I.$$

Combining this lower eigenvalue bound with the upper determinant bound in (3.8), we obtain

$$M_{11}(x_0) \leq C.$$

Since G attains its maximum at (x_0, e_1), for every $(x, \xi) \in \mathbb{T}^n \times \mathbb{S}^{n-1}$,

$$\log M_{\xi\xi}(x) - B\psi(x) \leq \log M_{11}(x_0) - B\psi(x_0).$$

The C^0 estimate for ψ therefore gives

$$M_{\xi\xi}(x) \leq C.$$

Thus

$$M \leq CI \quad \text{on } \mathbb{T}^n.$$

Finally, the lower determinant bound in (3.8) and the upper eigenvalue bound imply

$$M \geq C^{-1}I.$$

This proves (3.9). □

Standard regularity theory now gives higher-order estimates.

Proposition 3.8 (Higher-order estimates)**.** *For every integer $k \geq 0$, there exists a constant*

$$C_k = C_k\big(A, a, n, \|V\|_{C^{k+2}(\mathbb{T}^n)}\big)$$

such that every zero-average solution of (3.2) *satisfies*

$$\|\psi\|_{C^k(\mathbb{T}^n)} + |c| \leq C_k. \tag{3.12}$$

Proof. Proposition 3.7 gives uniform ellipticity:

$$C^{-1}I \le A + D^2\psi \le CI.$$

Moreover, $D^2\psi$ is uniformly bounded, so $D\psi$ is uniformly Lipschitz. Hence the right-hand side of

$$\log\det(A + D^2\psi) = -ta \cdot D\psi + tV - c$$

is uniformly bounded in $C^\alpha(\mathbb{T}^n)$ for every fixed $\alpha \in (0,1)$. The Evans–Krylov estimate for concave uniformly elliptic equations [Eva82, Kry83] therefore gives

$$\|\psi\|_{C^{2,\alpha}(\mathbb{T}^n)} \le C.$$

Differentiating the equation produces uniformly elliptic linear equations for the derivatives of ψ. Standard Schauder estimates and induction yield (3.12). □

The continuity method. We complete the continuity argument.

Proof of Theorem 1.2. Let $\mathcal{S} \subset [0,1]$ be the set of parameters t for which there exists a smooth pair (ψ, c) solving (3.2), with $A + D^2\psi > 0$ and

$$\int_{\mathbb{T}^n} \psi = 0.$$

At $t = 0$, the pair

$$\psi = 0, \quad c = -\log\det A$$

solves (3.2). Hence

$$0 \in \mathcal{S}.$$

We next prove openness. Fix $t_0 \in \mathcal{S}$, and let (ψ, c) be the corresponding solution. Set

$$M = A + D^2\psi.$$

The linearization of the left-hand side of (3.2) in the variables (ψ, c) is

$$(\eta, \kappa) \longmapsto L\eta + \kappa, \quad L\eta = M^{ij}\eta_{ij} + t_0 a \cdot D\eta. \tag{3.13}$$

We view this as a map

$$C^{2,\alpha}_0(\mathbb{T}^n) \times \mathbb{R} \longrightarrow C^\alpha(\mathbb{T}^n),$$

where

$$C^{2,\alpha}_0(\mathbb{T}^n) = \left\{ \eta \in C^{2,\alpha}(\mathbb{T}^n) : \int_{\mathbb{T}^n} \eta = 0 \right\}.$$

The strong maximum principle gives

$$\ker L = \{\text{constants}\}.$$

As an elliptic operator on the compact manifold $\mathbb{T}^n$, L is Fredholm as a map

$$C^{2,\alpha}(\mathbb{T}^n) \longrightarrow C^\alpha(\mathbb{T}^n).$$

Its index is zero. Indeed, the lower-order drift can first be removed through the elliptic homotopy

$$L_s\eta = M^{ij}\eta_{ij} + s\, t_0 a \cdot D\eta, \qquad 0 \le s \le 1,$$

and the positive definite principal coefficients can then be deformed through elliptic operators to those of the flat Laplacian. The Fredholm index is constant along these homotopies, while the Laplacian on $\mathbb{T}^n$ has index zero. It follows that

$$\operatorname{codim} \operatorname{Ran} L = 1.$$

Furthermore,

$$1 \notin \operatorname{Ran} L.$$

Indeed, if $L\eta = 1$, then at a maximum point of η, $L\eta \le 0$, which is impossible. Hence

$$C^\alpha(\mathbb{T}^n) = \operatorname{Ran} L \oplus \operatorname{span}\{1\}.$$

Adding a constant to η does not change $L\eta$, so restricting the domain of L to the zero-average subspace does not change its range. Therefore the map in (3.13) is surjective.

It is also injective. If

$$L\eta + \kappa = 0$$

and η has zero average, then $\kappa > 0$ is impossible by evaluating at a minimum point of η, while $\kappa < 0$ is impossible by evaluating at a maximum point. Thus $\kappa = 0$, and the strong maximum principle gives that η is constant. Its zero average then implies $\eta = 0$.

Consequently,

$$(\eta, \kappa) \longmapsto L\eta + \kappa$$

is an isomorphism. The implicit function theorem proves that $\mathcal{S}$ is open.

The *a priori* estimates in Lemmas 3.5 and Proposition 3.8 imply that $\mathcal{S}$ is closed. Indeed, if

$$t_j \in \mathcal{S}, \quad t_j \to t_\infty,$$

then the corresponding solutions admit uniform C^k bounds for every k. A diagonal compactness argument yields a smooth limiting solution at t_∞.

Thus $\mathcal{S} = [0,1]$. At $t = 1$, we obtain a smooth pair (ψ_A, c_A) satisfying (3.1). Lemma 3.3 gives uniqueness of c_A, and uniqueness of the zero-average solution ψ_A. $\square$

4. Periodic correctors and bounded-corrector rigidity

Throughout this section, $A = A^\top > 0$ is fixed, and (ψ_A, c_A) denotes the unique zero-average solution of the cell problem

$$\det(A + D^2\psi_A) = \exp\{-a \cdot D\psi_A + V - c_A\}, \quad A + D^2\psi_A > 0 \quad \text{on } \mathbb{T}^n. \tag{4.1}$$

We extend V and ψ_A periodically to $\mathbb{R}^n$, and write

$$M_A = A + D^2\psi_A, \quad \Phi_A(x) = \frac{1}{2}x^\top Ax + \psi_A(x).$$

Thus

$$D^2\Phi_A = M_A.$$

We study entire solutions of

$$\det D^2u = \exp\{-a \cdot Du + b \cdot x + V(x) - c_0\} \quad \text{in } \mathbb{R}^n. \tag{4.2}$$

We first classify exactly quadratic-periodic solutions, then derive the compatibility condition $b = Aa$ from weaker asymptotic assumptions, and finally prove the bounded-corrector Liouville theorem.

Theorem 4.1 (Classification of quadratic-periodic solutions). *Let $A = A^\top > 0$, $\ell \in \mathbb{R}^n$, and $\psi \in C^\infty(\mathbb{T}^n)$. Suppose that*

$$A + D^2\psi > 0.$$

Then

$$u(x) = \frac{1}{2}x^\top Ax + \ell \cdot x + \psi(x)$$

solves (4.2) if and only if

$$b = Aa, \quad c_0 + a \cdot \ell = c_A, \quad \psi = \psi_A + C$$

for some constant $C \in \mathbb{R}$.

In particular, if $b = Aa$, then

$$u(x) = \frac{1}{2}x^\top Ax + \ell \cdot x + \psi_A(x) + C$$

is a solution precisely when $c_0 + a \cdot \ell = c_A$, with $C \in \mathbb{R}$ arbitrary.

Proof. For

$$u(x) = \frac{1}{2}x^\top Ax + \ell \cdot x + \psi(x),$$

we have

$$Du = Ax + \ell + D\psi, \quad D^2u = A + D^2\psi.$$

Since A is symmetric,

$$-a \cdot Du + b \cdot x + V - c_0 = (b - Aa) \cdot x - a \cdot \ell - a \cdot D\psi + V - c_0.$$

Hence the equation becomes

$$\det(A + D^2\psi) = \exp\{(b - Aa) \cdot x - a \cdot D\psi + V - (c_0 + a \cdot \ell)\}. \tag{4.3}$$

The left-hand side of (4.3) is $\mathbb{Z}^n$-periodic. The functions $D\psi$ and V are also $\mathbb{Z}^n$-periodic. Set

$$m = b - Aa.$$

If $m \neq 0$, then some coordinate m_i is nonzero. Taking $k = e_i \in \mathbb{Z}^n$ and comparing (4.3) at x and $x + k$ gives

$$e^{m \cdot k} = 1.$$

This is impossible because $m \cdot k = m_i \neq 0$. Therefore

$$b = Aa.$$

With this compatibility imposed, (4.3) reduces to

$$\det(A + D^2\psi) = \exp\{-a \cdot D\psi + V - (c_0 + a \cdot \ell)\}.$$

The uniqueness of the cell problem, without imposing the additive normalization on ψ, gives

$$c_0 + a \cdot \ell = c_A$$

and

$$\psi = \psi_A + C$$

for some constant C. Conversely, these conditions reduce the full-space equation exactly to (4.1). The additive constant does not occur in the equation. □

Remark 4.2. If $a \neq 0$ and $b = Aa$, then for every $c_0 \in \mathbb{R}$ one can choose $\ell \in \mathbb{R}^n$ so that

$$a \cdot \ell = c_A - c_0.$$

If $a = 0$, then $b = Aa$ is equivalent to $b = 0$, and the condition on the normalizing constant reduces to

$$c_0 = c_A.$$

We next show that $b = Aa$ is not merely a consequence of exact periodicity. It is forced whenever the perturbation of the quadratic polynomial has sublinear gradient growth.

We begin with two elementary consequences of global semiconvexity.

Lemma 4.3 (Gradient bounds for globally semiconvex functions)**.** *Let $f \in C^2(\mathbb{R}^n)$ satisfy*

$$D^2 f \geq -\Lambda I \quad \text{in } \mathbb{R}^n$$

for some $\Lambda \geq 0$.

(i) *If $f \in L^\infty(\mathbb{R}^n)$, then*

$$\|Df\|_{L^\infty(\mathbb{R}^n)} \leq \sqrt{2\Lambda \operatorname{osc}_{\mathbb{R}^n} f}. \tag{4.4}$$

When $\Lambda = 0$, the conclusion is understood as $Df \equiv 0$.

(ii) *If*

$$\sup_{B_R} |f| = o(R^2) \quad \text{as } R \to \infty, \tag{4.5}$$

then

$$\sup_{B_R} |Df| = o(R) \quad \text{as } R \to \infty. \tag{4.6}$$

Proof. Fix $x \in \mathbb{R}^n$ and $e \in \mathbb{S}^{n-1}$, and define

$$g(t) = f(x + te).$$

Then

$$g''(t) \geq -\Lambda.$$

Suppose first that f is bounded and

$$P = g'(0) > 0.$$

For $0 \leq t \leq P/\Lambda$, when $\Lambda > 0$,

$$g'(t) \geq P - \Lambda t.$$

Therefore

$$\begin{aligned}\operatorname{osc}_{\mathbb{R}^n} f &\geq g(P/\Lambda) - g(0)\\ &\geq \int_0^{P/\Lambda} (P - \Lambda t)\,\mathrm{d}t = \frac{P^2}{2\Lambda}.\end{aligned}$$

Thus

$$P \leq \sqrt{2\Lambda \operatorname{osc}_{\mathbb{R}^n} f}.$$

Applying the same argument in the direction $-e$ gives the corresponding lower bound for $g'(0)$. This proves (4.4). If $\Lambda = 0$, then f is bounded and convex, hence constant.

We now assume (4.5). Fix $\delta \in (0, 1)$. If $x \in B_R$, $e \in \mathbb{S}^{n-1}$, and $t = \delta R$, semiconvexity gives

$$f(x + te) \geq f(x) + tDf(x) \cdot e - \frac{\Lambda t^2}{2}$$

and

$$f(x - te) \geq f(x) - tDf(x) \cdot e - \frac{\Lambda t^2}{2}.$$

Since

$$x \pm te \in B_{(1+\delta)R},$$

we obtain

$$|Df(x) \cdot e| \leq \frac{2 \sup_{B_{(1+\delta)R}} |f|}{\delta R} + \frac{\Lambda \delta R}{2}.$$

Taking the supremum over $x \in B_R$ and $e \in \mathbb{S}^{n-1}$, dividing by R, and letting $R \to \infty$, we find

$$\limsup_{R\to\infty} \frac{\sup_{B_R} |Df|}{R} \leq \frac{\Lambda\delta}{2}.$$

Since $\delta > 0$ is arbitrary, this proves (4.6). □

We now prove the compatibility statement under the natural gradient-growth condition.

Proposition 4.4 (Compatibility under sublinear gradient growth)**.** *Let*

$$u(x) = \frac{1}{2} x^\top Ax + \ell \cdot x + f(x), \quad A = A^\top > 0,$$

be a smooth strictly convex solution of (4.2)*. Assume that*

$$\omega(R) := \sup_{B_R} |Df| = o(R) \quad \text{as } R \to \infty.$$

Then

$$b = Aa.$$

Proof. Set

$$m = b - Aa.$$

Since

$$Du = Ax + \ell + Df,$$

the equation becomes

$$\det D^2 u = \exp\{m \cdot x - a \cdot \ell - a \cdot Df + V(x) - c_0\}.$$

Suppose, for contradiction, that $m \neq 0$, and set

$$e = \frac{m}{|m|}.$$

For $R > 1$, define the spherical cap

$$E_R = \left\{ x \in B_R : e \cdot x \geq \frac{R}{2} \right\}.$$

There exists $c_n > 0$ such that

$$|E_R| \geq c_n R^n.$$

Moreover,

$$m \cdot x \geq \frac{|m|R}{2} \quad \text{on } E_R.$$

Since V is bounded and

$$|a \cdot Df| \leq |a|\omega(R) = o(R),$$

for all sufficiently large R we have, on E_R,

$$\log \det D^2 u \geq \frac{|m|R}{4} - C.$$

It follows that

$$\int_{B_R} \det D^2 u \, \mathrm{d}x \geq c R^n e^{|m|R/4}. \tag{4.7}$$

Because $D^2 u > 0$, the gradient map Du is injective. Indeed,

$$(Du(x) - Du(y)) \cdot (x - y) > 0 \quad \text{for } x \neq y.$$

Thus the multiplicity of Du on B_R is one, and the area formula gives

$$\int_{B_R} \det D^2 u \, \mathrm{d}x = |Du(B_R)|. \tag{4.8}$$

On the other hand,

$$Du(B_R) \subset AB_R + \ell + B_{\omega(R)}.$$

Since $\omega(R) = o(R)$, the set on the right is contained in a Euclidean ball of radius CR for all sufficiently large R. Hence

$$|Du(B_R)| \leq CR^n. \tag{4.9}$$

Estimates (4.7), (4.8), and (4.9) are incompatible. Therefore $m = 0$, which is precisely $b = Aa$. $\square$

Corollary 4.5 (Compatibility for subquadratic correctors)**.** *Let*

$$u(x) = \frac{1}{2}x^\top Ax + \ell \cdot x + f(x), \quad A = A^\top > 0,$$

be a smooth strictly convex solution of (4.2)*.*

If

$$\sup_{B_R} |f| = o(R^2) \quad \text{as } R \to \infty,$$

then

$$b = Aa.$$

Proof. The strict convexity condition gives

$$A + D^2 f > 0,$$

and hence

$$D^2 f \geq -\lambda_{\max}(A) I.$$

Lemma 4.3 gives

$$\sup_{B_R} |Df| = o(R),$$

so Proposition 4.4 applies. □

We next establish the compactness needed in the bounded-corrector Liouville argument.

Lemma 4.6 (Compactness of integer translates)**.** *Let*

$$h \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

satisfy

$$M_A + D^2 h > 0$$

and

$$\log\det(M_A + D^2 h) - \log\det M_A + a \cdot Dh = \delta \quad \text{in } \mathbb{R}^n \tag{4.10}$$

for some constant $\delta \in \mathbb{R}$. Then the family

$$\{h(\,\cdot\, + m) : m \in \mathbb{Z}^n\}$$

is precompact in $C^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Since M_A is smooth, periodic, and positive definite, there are constants

$$0 < \lambda_A \leq \Lambda_A < \infty$$

such that

$$\lambda_A I \leq M_A \leq \Lambda_A I \quad \text{on } \mathbb{R}^n. \tag{4.11}$$

The positivity of $M_A + D^2 h$ gives

$$D^2 h \geq -\Lambda_A I.$$

Lemma 4.3 and the boundedness of h then yield

$$\|Dh\|_{L^\infty(\mathbb{R}^n)} \leq L \tag{4.12}$$

for some finite constant L. Set also

$$H = \|h\|_{L^\infty(\mathbb{R}^n)}.$$

Equation (4.10) is equivalent to

$$\det(M_A + D^2 h) = e^{\delta - a\cdot Dh} \det M_A.$$

Using (4.11) and (4.12), we obtain constants

$$0 < \lambda \le \Lambda < \infty$$

such that

$$\lambda \le \det(M_A + D^2 h) \le \Lambda \quad \text{in } \mathbb{R}^n. \tag{4.13}$$

For $m \in \mathbb{Z}^n$, set

$$h_m(x) = h(x+m), \qquad U_m(x) = \Phi_A(x) + h_m(x).$$

The periodicity of M_A implies

$$D^2 U_m = M_A + D^2 h_m > 0,$$

and every h_m satisfies

$$\log\det(M_A + D^2 h_m) - \log\det M_A + a\cdot Dh_m = \delta. \tag{4.14}$$

Equivalently,

$$\det D^2 U_m = e^{\delta - a\cdot Dh_m} \det M_A. \tag{4.15}$$

The determinant bounds in (4.13) hold uniformly for all m.

We next establish uniform geometry for fixed-height sections. Fix $x_0 \in \mathbb{R}^n$, and define

$$\begin{aligned} q_{m,x_0}(z) &:= U_m(x_0+z) - U_m(x_0) - DU_m(x_0)\cdot z, \\ S_m(x_0,t) &:= \{x_0 + z : q_{m,x_0}(z) < t\}. \end{aligned}$$

By (4.11),

$$\frac{\lambda_A}{2}|z|^2 \le \Phi_A(x_0+z) - \Phi_A(x_0) - D\Phi_A(x_0)\cdot z \le \frac{\Lambda_A}{2}|z|^2.$$

Moreover, by the definitions of H and L,

$$|h_m(x_0+z) - h_m(x_0) - Dh_m(x_0)\cdot z| \le 2H + L|z|.$$

Consequently,

$$\frac{\lambda_A}{2}|z|^2 - L|z| - 2H \le q_{m,x_0}(z) \le \frac{\Lambda_A}{2}|z|^2 + L|z| + 2H. \tag{4.16}$$

All constants here are independent of m and x_0.

Choose

$$H_0 = 4\big(2\Lambda_A + 2L + 2H + 1\big).$$

The upper bound in (4.16) gives

$$B_2(x_0) \subset S_m\left(x_0, \frac{H_0}{4}\right). \tag{4.17}$$

Choose $R_0 > 2$ so large that

$$\frac{\lambda_A}{2} r^2 - Lr - 2H > H_0 \quad \text{for every } r \geq R_0.$$

The lower bound in (4.16) then gives

$$S_m(x_0, H_0) \subset B_{R_0}(x_0). \tag{4.18}$$

Thus all the sections $S_m(x_0, H_0)$ have uniform inner and outer ball control. We now make the normalization and the dependence of the constants explicit. Set

$$\Omega_{m,x_0} = S_m(x_0, H_0) - x_0.$$

By John's lemma, there is an affine map

$$\mathcal{A}_{m,x_0}(z) = B_{m,x_0} z + y_{m,x_0}$$

such that

$$B_1 \subset \mathcal{A}_{m,x_0}(\Omega_{m,x_0}) \subset B_n.$$

The ball inclusions (4.17) and (4.18) imply uniform bounds for B_{m,x_0}, B^{-1}_{m,x_0}, and $|\det B_{m,x_0}|^{\pm 1}$. Define on the normalized domain

$$\widetilde{\Omega}_{m,x_0} = \mathcal{A}_{m,x_0}(\Omega_{m,x_0})$$

the convex function

$$\widetilde{v}_{m,x_0}(y) = \frac{1}{H_0} q_{m,x_0}\big(\mathcal{A}^{-1}_{m,x_0}(y)\big) - 1.$$

Then $\widetilde{v}_{m,x_0} = 0$ on $\partial\widetilde{\Omega}_{m,x_0}$. Moreover, since $q_{m,x_0} \geq 0$ and $q_{m,x_0}(0) = 0$,

$$\min_{\widetilde{\Omega}_{m,x_0}} \widetilde{v}_{m,x_0} = \widetilde{v}_{m,x_0}\big(\mathcal{A}_{m,x_0}(0)\big) = -1.$$

Thus the normalized domains and potentials satisfy the standard normalization hypotheses with constants independent of m and x_0. Furthermore,

$$\det D^2 \widetilde{v}_{m,x_0}(y) = H_0^{-n} |\det B_{m,x_0}|^{-2} \det D^2 U_m\big(x_0 + \mathcal{A}^{-1}_{m,x_0}(y)\big).$$

Consequently, the normalized Monge–Ampère densities are bounded above and below by positive constants independent of m and x_0.

Caffarelli's interior $C^{1,\gamma}$ estimate on normalized sections [Caf90a, Caf91], in the formulation [Fig17, Theorem 4.20], now gives $\gamma \in (0,1)$ and a uniform constant C such that

$$[DU_m]_{C^\gamma(S_m(x_0, 3H_0/4))} \leq C.$$

Here and below the estimates are applied on fixed sublevel sets of the normalized potential; the constants therefore depend only on the fixed level ratios, the dimension, and the uniform determinant and normalization bounds. Since

$$Dh_m = DU_m - D\Phi_A$$

and Φ_A is fixed and smooth, the right-hand side

$$f_m(x) = e^{\delta - a \cdot Dh_m(x)} \det M_A(x)$$

of (4.15) is uniformly bounded in $C^\gamma(S_m(x_0, 3H_0/4))$ and is uniformly bounded away from zero. Moreover, after pulling the equation back by $\mathcal{A}_{m,x_0}$, the normalized right-hand sides have uniformly bounded C^γ norms. Indeed, the linear parts of $\mathcal{A}_{m,x_0}$ and their inverses are uniformly bounded.

Caffarelli's interior $C^{2,\alpha}$ estimate for Hölder densities [Caf90b, Caf91], in the formulation [Fig17, Theorem 4.42], , applied to the normalized equation and then transferred back to the original coordinates, yields some $\alpha \in (0, \gamma]$ such that

$$\|D^2 U_m\|_{C^\alpha(S_m(x_0, H_0/2))} \leq C.$$

The affine changes of variables above have uniformly bounded linear parts and inverses, so the constants remain independent of m and x_0. Since

$$B_2(x_0) \subset S_m\left(x_0, \frac{H_0}{4}\right) \subset S_m\left(x_0, \frac{H_0}{2}\right),$$

we obtain a uniform $C^{2,\alpha}$ bound on $B_2(x_0)$. Together with the determinant lower bound in (4.13), this makes

$$G_m := D^2 U_m = M_A + D^2 h_m$$

uniformly elliptic on $B_2(x_0)$.

For completeness, we spell out the first step of the Schauder bootstrap. Differentiating (4.14) in the x_k-direction gives

$$G_m^{ij} \partial_{ij}(h_m)_k + a \cdot D(h_m)_k = \left((M_A^{-1})^{ij} - G_m^{ij}\right) \partial_k (M_A)_{ij}. \tag{4.19}$$

The coefficients G_m^{ij} are uniformly elliptic and uniformly C^α, and the right-hand side is uniformly C^α, because M_A is fixed and smooth. Interior Schauder estimates applied to (4.19) give a uniform $C^{3,\alpha}$ bound for h_m on $B_{3/2}(x_0)$. Repeated differentiation and induction then yield, for every integer $r \geq 0$,

$$\|h_m\|_{C^r(B_1(x_0))} \leq C_r,$$

where C_r depends only on r, the fixed data, and finitely many norms of the fixed smooth matrix field M_A, but not on m or x_0. Hence the family $\{h_m\}$ is uniformly bounded in every C^r norm on compact subsets of $\mathbb{R}^n$. The Arzelà–Ascoli theorem and a diagonal argument give precompactness in $C^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. □

We can now prove that a bounded perturbation of the periodic cell solution is necessarily constant.

Theorem 4.7 (Bounded-corrector Liouville theorem)**.** *Let $h \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy*

$$M_A + D^2 h > 0$$

and

$$\log\det(M_A + D^2 h) - \log\det M_A + a \cdot Dh = \delta \quad \textit{in } \mathbb{R}^n$$

for some constant $\delta \in \mathbb{R}$. Then

$$\delta = 0 \quad \textit{and} \quad h = \text{constant}.$$

Proof. Fix $k \in \mathbb{Z}^n$, and define

$$C_k = \sup_{x \in \mathbb{R}^n} \big(h(x+k) - h(x) \big).$$

Since h is bounded, C_k is finite. Choose a sequence $x_j \in \mathbb{R}^n$ such that

$$h(x_j + k) - h(x_j) \to C_k.$$

Write

$$x_j = m_j + y_j, \quad m_j \in \mathbb{Z}^n, \quad y_j \in [0,1]^n.$$

After passing to a subsequence,

$$y_j \to y_\infty \in [0,1]^n.$$

By Lemma 4.6, after passing to a further subsequence,

$$h(\,\cdot\, + m_j) \to H \quad \text{in } C^\infty_{\mathrm{loc}}(\mathbb{R}^n),$$

where H is bounded and satisfies the same equation

$$\log\det(M_A + D^2 H) - \log\det M_A + a \cdot DH = \delta.$$

For every $x \in \mathbb{R}^n$,

$$h(x + m_j + k) - h(x + m_j) \le C_k.$$

Passing to the limit gives

$$H(x+k) - H(x) \le C_k \quad \text{for all } x \in \mathbb{R}^n. \tag{4.20}$$

The choice of the maximizing sequence gives equality at y_∞:

$$H(y_\infty + k) - H(y_\infty) = C_k. \tag{4.21}$$

Define

$$\widetilde{H}(x) = H(x+k) - C_k.$$

By (4.20),

$$\widetilde{H} \le H,$$

and by (4.21), equality holds at y_∞. Since $k \in \mathbb{Z}^n$ and M_A is periodic, the function $H(\,\cdot\, + k)$ satisfies the same equation as H. Subtracting the constant C_k also leaves the equation unchanged. Thus both $\widetilde{H}$ and H solve

$$\log\det(M_A + D^2 v) + a \cdot Dv = \log\det M_A + \delta.$$

The strong comparison principle, Lemma 3.2, yields

$$H(x+k) - C_k \equiv H(x).$$

Iterating,

$$H(x + qk) - H(x) = qC_k \quad \text{for every } q \in \mathbb{N}.$$

Since H is bounded, this is possible only if $C_k = 0$. Therefore

$$h(x+k) - h(x) \le 0 \quad \text{for all } x.$$

Applying the same argument to $-k$ and then replacing x by $x+k$ gives

$$h(x) - h(x+k) \le 0.$$

Hence

$$h(x+k) = h(x) \quad \text{for all } x \in \mathbb{R}^n, \quad k \in \mathbb{Z}^n.$$

Thus h is $\mathbb{Z}^n$-periodic.

At a maximum point x_+ of h,

$$Dh(x_+) = 0, \quad D^2h(x_+) \le 0.$$

Therefore

$$M_A(x_+) + D^2h(x_+) \le M_A(x_+),$$

and the equation gives $\delta \le 0$. At a minimum point x_-,

$$Dh(x_-) = 0, \quad D^2h(x_-) \ge 0,$$

so the same equation gives $\delta \ge 0$. Consequently, $\delta = 0$. It follows that $\psi_A + h$ and ψ_A solve the same periodic cell equation:

$$\log\det(A + D^2v) + a \cdot Dv = V - c_A.$$

The uniqueness statement from Section 3 implies that their difference h is constant. □

We finish the proof of the theorem stated in the Introduction.

Proof of Theorem 1.3. Write

$$f(x) = u(x) - \frac{1}{2}x^\top Ax - \ell \cdot x.$$

The compatibility conclusion under the subquadratic hypothesis follows from Corollary 4.5. Assume now that $f \in L^\infty(\mathbb{R}^n)$. Since a bounded function is subquadratic, we already know that $b = Aa$. Moreover,

$$f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

and

$$A + D^2f = D^2u > 0.$$

Set

$$h = f - \psi_A.$$

Since f is bounded and ψ_A is periodic, $h \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Moreover,

$$M_A + D^2h = A + D^2f = D^2u > 0.$$

Using $b = Aa$, the equation for u becomes

$$\log\det(A + D^2f) = -a \cdot Df + V - (c_0 + a \cdot \ell).$$

The cell equation gives

$$\log\det M_A = -a \cdot D\psi_A + V - c_A.$$

Subtracting these identities and using $Dh = Df - D\psi_A$, we obtain

$$\log\det(M_A + D^2h) - \log\det M_A + a \cdot Dh = c_A - (c_0 + a \cdot \ell).$$

Theorem 4.7 therefore implies

$$c_A - (c_0 + a \cdot \ell) = 0 \quad \text{and} \quad h = \text{constant}.$$

Consequently,

$$c_0 + a \cdot \ell = c_A \quad \text{and} \quad f = \psi_A + \text{constant}.$$

Equivalently,

$$u(x) = \frac{1}{2} x^\top A x + \ell \cdot x + \psi_A(x) + C$$

for some $C \in \mathbb{R}$. □

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